Notes on Rasmussen's Proof of Milnor's Conjecture

From last time we have

Definition. A cobordism between links L_0, L_1 is a smooth surface $W \subseteq S^3 \times [0,1]$ with $W|_{S^3\times\{0\}} = L_0$ and $W|_{S^3\times\{1\}} = L_1$ and boundary $\partial W = L_0 \cup L_1$.

We also recall the following:

- We may decompose a cobordism W into a series of elementary cobordisms corresponding to Reidemeister moves and Morse moves.
- A cobordism W induces a map $\Phi_W : H_{Lee}(L_0) \to H_{Lee}(L_1)$
- Two cobordisms are equivalent if we can relate them by a series of "movie moves". Theorem from last week: Φ_W is well-defined.

We will show that for knots Φ_W is an isomorphism, but first we introduce a related notion:

Definition. Two links L_0, L_1 are concordant if there exists an embedding $f : L_0 \times [0,1] \to S^3 \times [0,1]$ such that $f(L_0 \times \{i\}) = L_i \times \{i\}$ for $i \in [0,1]$. A knot K is called slice if K is concordant to the unknot.

More intuitively: Imprecisely: equivalent to saying that there exists a cobordism of genus 0 between L_0 and L_1 or L_0 and L_1 are connected by a cylinder in $S^3 \times [0,1]$ with boundary components L_i. With this formulation, a knot K is slice if it bounds a disk in B^4 (see example).

Fact: Concordance gives an equivalence relation on knots by setting $K_1 \sim K_2$ if $K_1 \# \overline{K_2}$ is slice. The set of knots modulo this relation forms an abelian group with the connect sum operation. This group is called the concordance group of knots in S^3 and is denoted $Conc(S^3)$.

Definition. The slice genus of a knot, K is the minimum genus g of a connected, orientable surface S (smoothly) embedded in B^4 that has K as a boundary.

In general, slice genus is difficult to compute. Lisa Piccirillo's proof that the Conway knot is not slice completed the classification of slice knots under 13 crossings only 2 years ago.

The following theorem provides a lower bound on slice genus

Theorem 1.

$$|s(K)| \leq 2g_*(K) \leq 2g(K)$$

where $g_*(K)$ denotes the slice genus.

To prove this theorem, we need to examine how the canonical generators of H_{Lee} behave under cobordism.

Proposition. Let W be a weakly connected cobordism from L_0 to L_1. Then $\Phi_W(z_{o_0})$ is a nonzero multiple of z_{o_1} where z_{o_i} denotes a canonical generator of $H_{Lee}(L_i)$ labeled by orientation o on L_i.

Definition. A cobordism W is weakly connected if all components of W have a boundary component in L_0.

The following corollary follows immediately from this proposition:

Corollary. If W is a connected cobordism between knots, then Φ_W is an isomorphism.

Proof. (of proposition) A cobordism W between two links L_1 and L_2 induces a morphism Φ_W on the Lee homology as follows:

For $W = W_1 \cup \cdots \cup W_k$, with corresponding links L_i, define $\Phi_W = \Phi_{W_1} \circ \cdots \circ \Phi_{W_k}$, where for each i, $\Phi_{W_i} : H_{Lee}(L_i) \to H_{Lee}(L_{i+1})$. We can then prove the proposition by induction on i. If W_i is an elementary cobordism corresponding to a Reidemeister move, we have (from our earlier discussion of Lee homology) a filtered map (of q-degree zero) ρ_* sending canonical generators to canonical generators. Therefore, it suffices to check that the proposition holds for an elementary cobordism corresponding to attaching a handle. For such W_i we apply $\iota', m'/\Delta'$, or ϵ' at each vertex of our cube of resolutions for $0, 1$, and 2-handle moves respectively. Whether we apply m' or Δ' depends on whether the 1-handle merges circles or splits circles at each vertex of the cube of resolutions (see example).

Divide the components of W_i into two sets:
Components of the 1st type have a boundary component in L_0
Components of the 2nd type have no boundary component in L_0

We call an orientation o on W_i permissible if it agrees with the orientation on S on components of the first type.

We will prove the slightly stronger statement that

$$\Phi_{W_i}(z_o) = \sum_I a_I z_{o_I}$$

for permissible orientations I of W_{i+1} arguing by induction on i.

- 0-handle: The generators of H_{Lee} of L_i are given by $z_o \otimes a$, $z_o \otimes b$, $z_{\overline{o}} \otimes a$ and $z_{\overline{o}} \otimes b$. The map $\Phi_W(z_o) = \iota(z_0) = z_0 \otimes \frac{1}{2}(a - b)$ is then a nonzero multiple of the two generators with the appropriate orientation.
- 1-handle: If O_I agrees with o_i then the two strands involved in the 1-handle move have opposite orientations, hence are both labeled a or both labeled b. We have

$$m(a \otimes a) = 2a \qquad \Delta(a) = a \otimes a$$
$$m(b \otimes b) = -2b \qquad \Delta(b) = b \otimes b$$

  So in this case, $\Phi_{W_i}(z_{o_i})$ is a nonzero multiple of z_{o_I}.

  If o_I is not compatible with o_i then we have two strands pointing in the same direction with different labels, so $\Phi_{W_i}(z_{o_i}) = 0$ because $m(a \otimes b) = m(b \otimes a) = 0$. From our discussion of Lee homology before, we know that we cannot split a circle into two circles with opposite labels, so we do not need to consider Δ' for this subcase.

  To complete the argument, we need to examine how the orientations behave under Φ_{W_i}. If attaching the 1-handle adds a component to L_i, then the number and type of components is the same, so the set of permissible orientations is preserved (naturally identified) If we merge two components, then we have three cases:

  - The merge only involves one component of W_i
  - The merge involves two components of W_i of the first type
  - The merge involves two components, one of which is of the second type

In the first two cases, the set of permissible orientations is preserved, as above. In the third case, the set of permissible orientations is halved, but we still preserve canonical generators.

- 2-handle: an orientation on W_i uniquely extends to an orientation on W_{i+1}. Then $\epsilon(a) = \epsilon(b) = 1$ implies that $\Phi_W(z_{o_i}) = z_{o_{i+1}}$

  Note that here, we might have two different orientations on W_{i+1} inducing two different orientations on L_{i+1}. But this can only happen when we have our two-handle capping a closed component and this violates the weakly connected hypothesis.

□

(See example)

Important note: We can compute the filtered $q-$degree of elementary cobordisms corresponding to handle attaching maps using our formula

$$q \text{ grading} = \text{degree}_A - |v| - \#\text{circles} - n_+ - 2n_-.$$

In our example, everything is preserved, except n_+, which decreases by 1. This holds in general and the degree of a 1-handle attaching is 1.

For 0-handle maps, we add a circle and everything else is preserved, so the q grading is -1. For $2-$handle maps, we add a delete a circle, but degree$_A$ decreases by 2 because we remove an x from the tensor product, so the q grading is -1 again. The filtered degree of an elementary cobordism corresponding to a handle attaching move is then the (negative) Euler characteristic, so the degree of the map Φ_W with our sign convention is

$$-\chi(W) = -(2 - 2g - 2) = 2g = -\#\text{0-handles } + \#\text{1-handles } - \#\text{2-handles}.$$

Now we wish to use Φ_W to get a relationship between $s(K)$ and slice genus of K in order to prove theorem 1.

Proof. (of Theorem 1) Let K be a knot with slice genus $g_*(K) = g$. Consider $x \neq 0$ with minimal grading in $H_{Lee}(K)$. We then have a cobordism W of genus g from K to the unknot U. By our corollary above, the induced map Φ_W is an isomorphism with $s_{min}(\Phi_W(x)) \geq -1 = s_{min}(U)$ (recall that for the unknot, $s(U) = 0$ and $s_{min}(U) = -1$. Since the $q-$degree of Φ_W is also $2g$, we have $s(x) \geq -2g - 1$ since Φ_W could have added at most $2g$ to the $q-$degree of x. We chose x to have minimal grading, so $s_{min}(K) \geq -2g - 1$. Hence, $s(K) \geq -2g$. The bound on $s(K)$ follows from the fact that $s(\overline{K}) = -s(K)$

□

Corollary. For a positive knot K, $s(K) = g_*(K) = g(K)$.

Proof. Last time we showed $g(k) \leq s(K)$ since a surface given by Seifert's algorithm has genus $s(K)$. With the inequality from the theorem above, we get the desired equality □

Corollary. (Milnor conjecture) The slice genus of a (p, q)-torus knot is $\frac{(p-1)(q-1)}{2}$.

Proof. The genus of the (p, q)-torus knot is known to be $\frac{(p-1)(q-1)}{2}$. Since torus knots are positive, the above corollary implies that $g_*(K) = g(K)$. □

We also have a nice algebraic property of the $s-$invariant. It behaves nicely with respect to connect sum and we have the following theorem:

Theorem 2. *The map s induces a homomorphism from $Conc(S^3)$ to $\mathbb{Z}$, where $conc(S^3)$ denotes the concordance group of knots in S^3*

Proof. See properties of $s(K)$ under connect sum for additivity (basic idea is that canonical generator of connect sum is tensor product of canonical generators of knots). To show that $s(K)$ is a well defined map, consider concordant knots K_1 and K_2. Then the connect sum is slice, so $0 = s(K_1 \# \overline{K_2}) = s(K_1) - s(K_2)$ and thus, $s(K_1) = s(K_2)$. □

(Cor: Knots differing by a single crossing satisfy $s(K_-) \leq s(K_+) \leq s(K_-) + 1$)