Example of Φ_W

From last time:

1. Defined Φ_W the map induced by a cobordism W between links and showed that it is an isomorphism between Lee homologies of knots.

2. Defined knot concordance and slice genus

3. Used Φ_W to show $|s(K)| \leq 2g_*(K) \leq 2g(K)$.

Today: example of Φ_W and cobordism between trefoil and unknot.

(see example)

We start by showing that a 0-smoothing corresponds to an $R1$ move, a 1-handle attaching map and an $R2$ move. This is more straightforward than our original 1-handle attaching map, as it allows for a way to systematically reduce the number of crossings and gives us an explicit map on the cube of resolutions. Using this reasoning, we see that we can get from the trefoil to the Hopf link, and from the Hopf link to the unknot by performing two 0-smoothings. These both correspond to a single 1-handle attaching map, the first of which splits our knot into two components, and the second of which joins the two components. Therefore, we get a genus one cobordism between the trefoil and the unknot.

On the level of the complex, when we fix a crossing of the diagram of L and do a 0-smoothing to get a new diagram L_0, we have that the resulting complex is a subcomplex of the original, i.e., $C(L_0) \subset C(L)$. The induced map is then just the projection $C(L) \to C(L_0)$. In our example, we can see that the canonical generators at the first vertex of our cube of resolutions get sent to the same generators in our second complex. The Hopf link has four canonical generators, so this map is injective, but not an isomorphism. Then projecting down to the subcomplex associated to the unknot, we retain our two original canonical generators, giving us the isomorphism from the proposition.

We can compute the $q-$degree of the map quite easily: Since $deg_A, |v|, \#$ of circles , and n_- are all under this quotient map and n_+ decreases by one, we have

$$q \text{ grading} = \text{degree}_A - |v| - \#\text{circles} - n_+ - 2n_-$$

increases by one. This is the same as our original 1-handle attaching map since Reidemeister moves don't affect $q-$degree.

We can also quickly compute the $q-$degree of the 0-handle and 2-handle attaching maps. For 0-handle maps, we add a circle and everything else is preserved, so the q grading is -1. For 2$-$handle maps, we add a delete a circle, but degree$_A$ decreases by 2 because we remove an x from the tensor product, so the q grading is -1 again. The filtered degree of an elementary cobordism corresponding to a handle attaching move is then the (negative) Euler characteristic, so the degree of the map Φ_W with our sign convention is

$$-\#0\text{-handles } + \#1\text{-handles } - \#2\text{-handles} = -\chi(W).$$

That's what we used to prove our theorem that $s(K)$ was a lower bound on slice genus.

With this theorem, we get two immediate corollaries:

Corollary. For a positive knot K, $s(K) = g_*(K) = g(K)$.

Proof. Two weeks ago, we showed $g(k) \leq s(K)$ since a surface given by Seifert's algorithm has genus $s(K)$. With the bound from the theorem above, we get the desired equality. $\square$

Corollary. (Milnor conjecture) The slice genus of a (p, q)-torus knot is $\frac{(p-1)(q-1)}{2}$.

Proof. The genus of the (p, q)-torus knot is known to be $\frac{(p-1)(q-1)}{2}$. Since torus knots are positive, the above corollary implies that $g_*(K) = g(K)$. $\square$