S-invariant

Tuesday, April 7, 2020 10:28 AM

__Last time:__

__Thm__ There is a spectral sequence

$E_1 =$ Khovanov homology

$E_\infty =$ Lee homology

$d_{Lee} = d_{Kh} + \Phi$, then

$d_0 = d_{Kh}$, $d_1 = \Phi$, d_r has bidegree

$(0,1)$ $\quad$ $(-4,1)$ $\quad$ $(q,t) = (-4r, 1)$.

__More on gradings__

Recall: we defined

$-n_+ + 2n_-$

$$\deg_q = \deg_A - \#\text{circles} - |v| + \text{shift}$$

$$\deg_t = |v| + \text{shift}$$

$-n_-$

where $n_+, n_- = \#$ positive/negative crossings.

__Rmk 1__ With these shifts, Khovanov homology is a bigraded invariant of links.

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Rmk 2 Bar-Natan and Rasmussen use opposite grading $A = \langle 1, x \rangle$, $1 = v_+$, $x = v_-$

$p(v_+) = 1 \qquad p(v_-) = -1$

on $A^{\otimes k}$ $\quad p(v_\pm \otimes \dots \otimes v_\pm) =$

$= p(v_\pm) + \dots + p(v_\pm)$

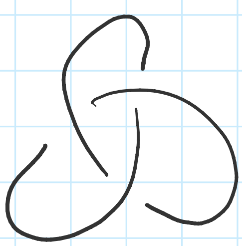
$= \#\text{circles} - \deg_A$, $\#\text{circles} = k$

(for each copy of A, $p = 1 - \deg_A$)

BN, Rasmussen: $\deg_q = p + \underbrace{|v| - n_-}_{\text{homol.}} + n_+ - n_-$

$= \#\text{circles} - \deg_A + |v| + n_+ - 2n_-$

$= (-1)\cdot(\text{our } q\text{-degree})$.



$$\begin{tikzcd}
|v|=0 & |v|=1 & |v|=2 & |v|=3 \\
 & A \arrow[r] \arrow[dr] & A\otimes A \arrow[dr] & \\
A\otimes A \arrow[ur] \arrow[r] \arrow[dr] & A \arrow[ur] \arrow[dr] & A\otimes A \arrow[r] & A\otimes A\otimes A \\
 & A \arrow[ur] \arrow[r] & A\otimes A \arrow[ur] &
\end{tikzcd}$$

$n_+ = 3$

$n_- = 0$ $\quad 1\otimes x - x\otimes 1$

$n_- = 0$ $\quad 1\otimes x - x\otimes 1$, $\; x\otimes x$ in $\ker(m)$

$\deg_A = 2$ ①

$\deg_A = 4$ ②

one more class

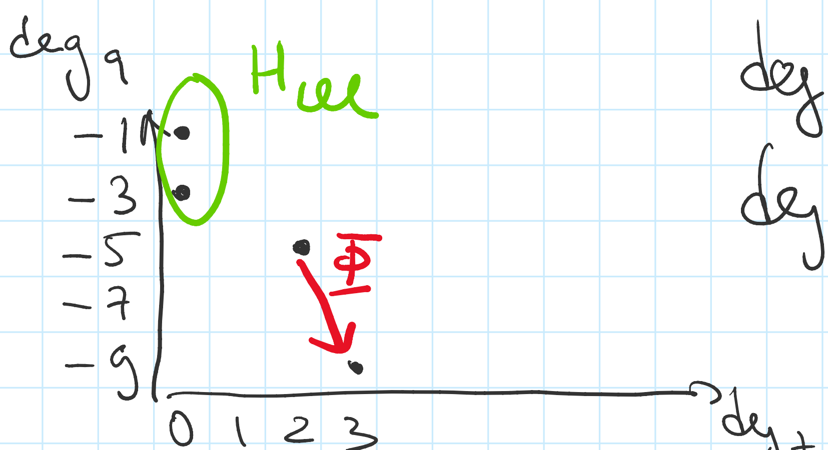
$\mathrm{coker}(\mu) = 1\otimes 1\otimes 1$

$\deg_A = 0$ ④

in span $(1\otimes x, 0, 0)$, $(x\otimes 1, 0, 0)$, $(0, 1\otimes x, 0)$ ...

$\deg_A = 2$ ③

	①	②	③	4
$\deg_A$	2	4	2	0
# circ	2	2	2	3
$\lvert v\rvert$	0	0	2	3
n_+	3	3	3	3
n_-	0	0	0	0
$\deg_A - \#c - \lvert v\rvert - n_+ + 2n_-$	−3	−1	−5	−9



$$\deg(\Phi) = (\overset{q}{-4}, \overset{t}{1})$$

$$\deg(d_r) = (-4r, 1)$$

Know: $\dim(H_{ee}) = 2$ for any knot!

$H_{ee} \,\|\, E_\infty$

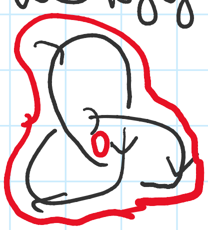
Φ is sometimes called "knight move"

φ is sometimes called knight's move

Rarely there are higher differentials.

<u>Rmk</u> For knots, both generators of Lee homology are in the same homological degree.

<u>Proof</u> Recall that the generators of Lee homology come from oriented resolutions.



If we change orientation, resolution is the same.

<u>Thm 1</u> (Rasmussen) The two surviving generators on E_∞ page of Lee spectral sequence have q-degrees different by 2: s_{max}, s_{min}

Define $\boxed{s = \dfrac{s_{max} + s_{min}}{2} = s\text{-invariant of a knot}}$

For trefoil, $s_{max} = -1$, $s_{min} = -3$

$\Rightarrow s = -2$.

We will prove it today.

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<u>Thm 2 (Rasmussen)</u> If Σ = genus g surface in B^4 boundary knot K, then

$$\boxed{2g(K) \geq |s(K)|}$$

<u>Thm 2: will prove later.</u>

Can define s_{max}, s_{min} without spectral sequences:

$[x] = H_{Lee}$, define

$s(x) = \max\{ \deg_q(z) : z \sim x$ in Lee complex$\}$

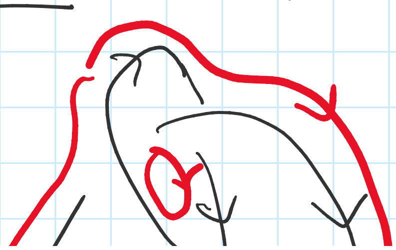
in Lee/Khovanov complex

(note: Lee differential does not preserve q-grading $\Rightarrow$ Lee complex is filtered).

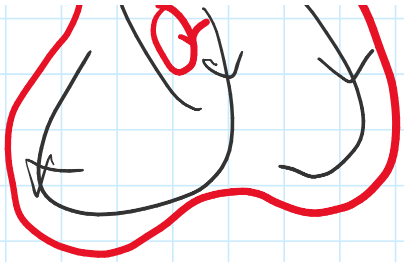
$$s_{max} = \max_{x \in H_{Lee}} s(x)$$

$$s_{min} = \min_{x \in H_{Lee}} s(x).$$

<u>Ex:</u> Trefoil. Generators of Lee homology come from oriented resolutions



- label each circle by a



- label each circle by a or b where $a = 1 + X$, $b = 1 - X$

Here: 2 generators in Lee homology

$$a \otimes b = (1+X) \otimes (1-X) = 1 \otimes 1 - 1 \otimes X + X \otimes 1 - \underline{X \otimes X}$$

$$b \otimes a = (1-X) \otimes (1+X) = 1 \otimes 1 + 1 \otimes X - X \otimes 1 - \underline{X \otimes X}$$

max $\deg_A = 4$

$$a \otimes b + b \otimes a = 2(1 \otimes 1 - X \otimes X) \quad \max \deg_A = 4$$

$$a \otimes b - b \otimes a = 2(X \otimes 1 - 1 \otimes X) \quad \max \deg_A = 2$$

Even if Lee generators are canonical, we may need to take their linear combination to find s_{max}, s_{min}.

<u>Lemma</u> $s_{max} \neq s_{min}$, moreover, they have different remainders mod 4.

<u>Proof</u>: First, observe that $\deg_q(\Phi)$ and $\deg_q(d_r)$ is divisible by 4 $\Rightarrow$ spectral sequence breaks into pieces in each remainder mod 4

We need to prove that in two remainders we have nontrivial classes in Lee homology.

We have nontrivial classes in Lee homology.

Consider an involution on A: $i(1) = 1 \leftarrow \deg_A = 0$
$i(x) = -x \quad \deg_A = 2$

On $A \otimes A \ldots \otimes A$: $i(x_1 \otimes \ldots \otimes x_k) =$

$$\begin{cases} x_1 \otimes \ldots \otimes x_k & \text{if } \deg_A(-) = 0 \bmod 4 \\ -x_1 \ldots \otimes x_k & \text{if } \deg_A(-) = 2 \bmod 4 \end{cases}$$

$i(a) = b \qquad i(b) = a$

If z_1, z_2 are Lee's canonical generators then $i(z_1) = z_2$, $i(z_2) = z_1$,

Therefore $z_1 + z_2$, $z_1 - z_2$ are fixed by this involution i, and so supported in same remainder mod 4.

That is, all terms in $z_1 + z_2$ are in same remainder mod 4

all terms in $z_1 - z_2$ in other remainder.

All terms in $z_1 + z_2$ have $\deg_A = 0 \bmod 4$ since $i(z_1 + z_2) = z_1 + z_2$

All terms in $z_1 - z_2$ have $\deg_A = 2 \bmod 4$ since $i(z_1 - z_2) = -(z_1 - z_2)$.

$\deg_{h} = \deg_A + \ldots$

$\deg_q = \deg_A + \dots$

Rmk Filtered complex

$\subset \mathcal{F}^i \subset \mathcal{F}^{i+1} \subset \mathcal{F}^{i+2} \subset \dots \subset \mathcal{C}$

where all $\mathcal{F}^i$ are subcomplexes of $\mathcal{C}$

$d(\mathcal{F}^i) \subset \mathcal{F}^i$

In our case, Lee complex is filtered,

$\mathcal{F}^i = \text{Span}\big(\text{all generators with } \deg_q \leq i\big)$

$d_{Lee} = \underset{\substack{q \\ \text{preserves} \\ q\text{-deg}}}{d} + \Phi$ decreases q-deg by 4

x = sum of generators with $\deg_q \leq i$

$d(x)$ = sum of generators with $\deg_q \leq i$

$\Phi(x)$ = sum of generators with $\deg_q \leq i-4$

$d_{Lee}(x)$ = sum of generators with $\deg_q \leq i$

$\Rightarrow d_{Lee}(\mathcal{F}^i) \subset \mathcal{F}^i$.

$s(x) = \max\{ i : \text{there is a class in } \mathcal{F}^i \text{ representing } x \}$

F^i representing x $\}$.