Tuesday, February 18, 2020

Define the complex

$$\cdots \longrightarrow C_i \xrightarrow{d_i} C_{i+1} \xrightarrow{d_{i+1}} C_{i+2} \longrightarrow \cdots$$

where $H_i = \text{Ker}(d_i)/\text{Ker}(d_{i-1})$

Claim: $\sum (-1)^i \dim C_i = \sum (-1)^i \dim H_i$

$= \text{Euler Characteristic}$

proof: $\dim C_i = \dim \ker d_i + \dim \text{Im}\, d_i$

$$\sum (-1)^i \dim C_i = \sum (-1)^i \dim \ker d_i + \sum (-1)^i \dim \text{Im}\, d_i$$
$$= \sum (-1)^i \dim \ker d_i - \sum (-1)^i \dim \text{Im}\, d_{i-1}$$
$$= \sum (-1)^i \dim H_i \quad \blacksquare$$

Computation of Khovanov Homology for right-handed trefoil

$$\begin{tikzcd}
 & A \arrow[r, "d_1"] \arrow[dr] & A^2 \arrow[dr, "d_2"] & \\
A^2 \arrow[ur, "d_0"] \arrow[r] \arrow[dr] & A \arrow[ur] \arrow[dr] & A^2 \arrow[r] & A^3 \\
 & A \arrow[ur] \arrow[r] & A^2 \arrow[ur] &
\end{tikzcd}$$

$H_0 = \text{Ker}(m) = \langle x \otimes x, 1 \otimes x - x \otimes 1 \rangle$

$H_1 : \text{Ker}(d_1) = (1,1,1) = \text{Im}(d_0) \Rightarrow H_1 = 0$

(x,x,x)

$H_3 : \text{Im}(d_2) :$

$(1 \otimes x + x \otimes 1) \otimes 1 = \alpha \qquad 1 \otimes 1 \otimes x + x \otimes 1 \otimes 1 = \beta$

$(1 \otimes x + x \otimes 1) \otimes x \qquad 1 \otimes x \otimes x + x \otimes x \otimes 1$

$\boxed{(x \otimes x) \otimes 1} \qquad \boxed{x \otimes 1 \otimes x}$

$\boxed{x \otimes x \otimes x} \qquad x \otimes x \otimes x$

$1 \otimes 1 \otimes x + 1 \otimes x \otimes 1 = \gamma$

$x \otimes 1 \otimes x + x \otimes x \otimes 1$

$\boxed{1 \otimes x \otimes x}$

$x \otimes x \otimes x \qquad \Rightarrow H_3 = \langle 1 \otimes 1 \otimes 1 \rangle$

NOTE: $\alpha - \beta + \gamma = 2(1 \otimes x \otimes 1)$

NOTE: $H_3 = \langle 1 \otimes 1 \otimes 1 \rangle \otimes \mathbb{Z}_2$ if working over $\mathbb{Z}$

Using Euler Characteristic

$$\chi = 4 - 6 + 12 - 8 = 2$$

$$= \underbrace{\dim H_0}_{2} - \underbrace{\dim H_1}_{0} + \dim H_2 - \underbrace{\dim H_3}_{1}$$

$$\Rightarrow \dim H_2 = 1$$

$$\begin{tikzcd}
 & & 1 \otimes x \arrow[dddr] \arrow[ddddr] & \\
 & 1 \arrow[ur] \arrow[r] & x \otimes 1 & \\
 & & & x \otimes x \otimes 1 \\
1 \otimes 1 \arrow[uur] \arrow[r] \arrow[ddr] & 1 \arrow[uuur] \arrow[uur] \arrow[dddr] \arrow[ddddr] & 1 \otimes x & x \otimes 1 \otimes x \\
 & & x \otimes 1 & 1 \otimes x \otimes x \\
 & 1 \arrow[uur] \arrow[ur] & & \\
 & & 1 \otimes x & \\
 & & x \otimes 1 &
\end{tikzcd}$$

$$\chi = 1 - 3 + 6 - 3 = 1$$

H_* in this q-degree: 0 0 ? 0] ?

Q: How to get polynomial from homology?

$H = \bigoplus H^{i,j}$ where $i =$ q-degree

$j =$ Hom. degree

then $\sum \dim H^{ij} q^i t^j$

Lemma: $H^{i,j}$ homology in q-degree i

homological degree j

$$\sum \dim H^{i,j} q^i (-1)^j = \text{Jones polynomial}$$

proof: $\sum \dim H^{i,j} (-1)^j q^i = \sum \dim C^{i,j} (-1)^j q^i$

q-grading = $\deg_A$ - #circles - $|v|$

$$A^{\otimes k} \longrightarrow (\underset{\overset{\nearrow}{1}}{1}+\underset{\overset{\nwarrow}{x}}{q^2})^k \cdot \underbrace{q^{-k}}_{\#\text{circles}} q^{-|v|} = (q+q^{-1})^k \cdot q^{-|v|}$$

e.g. $A \longrightarrow (1+q^2)$

$A \otimes A \longrightarrow (1+q^2)(1+q^2)$

$$\sum \dim C^{ij} (-1)^j q^i = \sum_v (-1)^{|v|} (q+q^{-1})^k \cdot q^{-|v|}$$
$$= \sum_v (-q)^{|v|} (q+q^{-1})^{\#\text{circles}} \qquad \blacksquare$$

Methods to simplify Khovanov Homology Computation:

$$\begin{tikzcd}
\cdots \arrow[r] & C_i \arrow[r] \arrow[d, phantom, "\cup" description] & C_{i+1} \arrow[r] \arrow[d, phantom, "\cup" description] & C_{i+2} \arrow[r] \arrow[d, phantom, "\cup" description] & \cdots \\
 & A_i & A_{i+1} & A_{i+2} &
\end{tikzcd}$$

- (A_i) is a <u>subcomplex</u> if $d(A_i) \subset A_{i+1}$
- Quotient complex $\cdots \to (C_i/A_i) \to (C_{i+1}/A_{i+1}) \to \cdots$

NOTE: $0 \to A. \to C. \to B. \to 0$
where $B_i = C_i/A_i$ is a short exact sequence of complexes

Key fact: long exact sequences

$$\cdots \to H_i(A.) \to H_i(C.) \to H_i(B.) \to H_{i+1}(A.) \to H_{i+1}(C.) \to H_{i+1}(B.) \to \cdots$$

Cor: a) If $H_i(A.) = 0$ $\forall i$ then $H_i(C.) = H_i(B.)$
$\to A.$ is acyclic

b) If $H_i(B.) = 0$ $\forall i$ then $H_i(C.) = H_i(A.)$
$\to B.$ is acyclic

a)
$$\begin{tikzcd}
{} \arrow[r] & C_i \arrow[r] \arrow[rdd] & C_{i+1} \arrow[r] \arrow[rdd] & C_{i+2} \arrow[r] & C_{i+3} \arrow[r] & \cdots \\
 & & \oplus & \oplus & & \\
 & & \mathbb{C} \arrow[r, "1"] & \mathbb{C} & &
\end{tikzcd}$$
subcomplex $A.$

has same homology as

$$\to C_i \to C_{i+1} \to C_{i+2} \to C_{i+3}$$

b)

$$\begin{tikzcd}
\arrow[r] & C_i \arrow[r] & C_{i+1} \arrow[r] & C_{i+2} \arrow[r] & C_{i+3} \\
 & & \oplus & \oplus & \\
 & & \mathbb{C} \arrow[r, "1"] \arrow[uur] & \mathbb{C} \arrow[uur] &
\end{tikzcd}$$

Subcomplex A

quotient

Example:

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dr, "1"] & \\
\mathbb{C} \arrow[ur, "1"] \arrow[dr, "1"'] & & \mathbb{C} \\
 & \mathbb{C} \arrow[ur, "-1"'] &
\end{tikzcd}$$

Subcomplex

<u>Hopf Link</u>

$$\begin{tikzcd}
 & A \arrow[dr] & \\
A \otimes A \arrow[ur] \arrow[dr] & & A \otimes A \\
 & A \arrow[ur] &
\end{tikzcd}$$

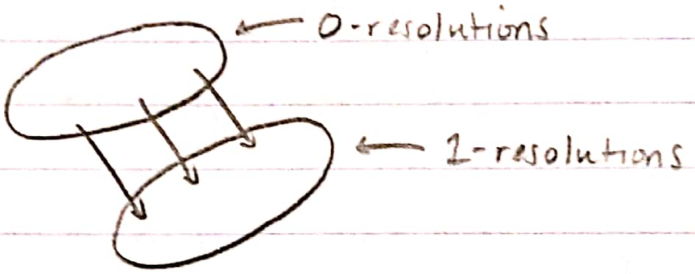
$$\begin{tikzcd}
 & & 1 \otimes 1 \\
 & 1 \arrow[dr] \arrow[ddr] & \\
1 \otimes 1 \arrow[ur] \arrow[ddr] & & 1 \otimes x \\
 & & x \otimes 1 \\
 & 1 \arrow[uur] \arrow[ur] & \\
 & x \arrow[dr] & \\
1 \otimes x \arrow[ur] \arrow[ddr] & & x \otimes x \\
x \otimes 1 \arrow[uur] \arrow[dr] & & \\
 & x \arrow[uur] & \\
x \otimes x & &
\end{tikzcd}$$

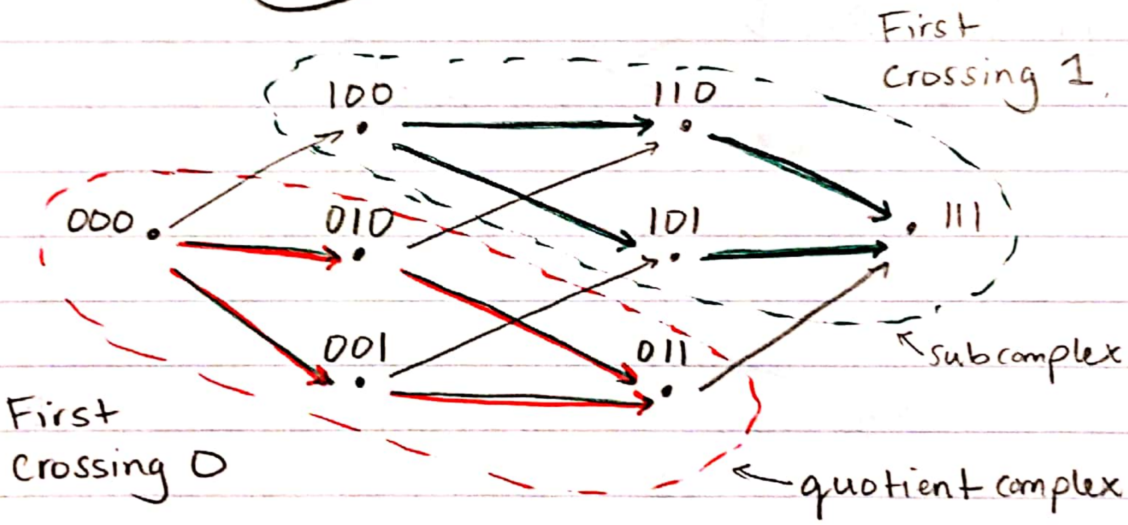
acyclic quotient

Fix a crossing in the diagram

Q: What does $\times =)(- q \asymp$ mean for K.H.?

- Cube of resolutions is sliced into two (n-1)-dim subcubes





0

↓

Khovanov complex for 1-resolution

↓

Khovanov complex for a diagram

↓

Khovanov complex for 0-resolution

↓

0