Tuesday, February 25, 2020

Lee Homology

Let $V = \langle 1, x \rangle$

For Khovanov homology: $\mathbb{Q}/(x^2)$

$m: V \otimes V \to V$

$1 \otimes 1 \to 1$

$1 \otimes x \to x$

$x \otimes 1 \to x$

$x \otimes x \to 0$

$\Delta: V \to V \otimes V$

$1 \to 1 \otimes x + x \otimes 1$

$x \to x \otimes x$

For Lee homology: $\mathbb{Q}/(x^2-1)$

$m: V \otimes V \to V$

$1 \otimes 1 \to 1$

$1 \otimes x \to x$

$x \otimes 1 \to x$

$x \otimes x \to 1$

$\Delta: V \to V \otimes V$

$1 \to 1 \otimes x + x \otimes 1$

$x \to x \otimes x + 1 \otimes 1$

NOTE: Lee homology carries more symmetry than Khovanov

Let $a = 1 + x$, $b = x - 1$ be a basis for V then

$m'(a \otimes a) = m'((1+x) \otimes (1+x)) = 1 + x + x + 1 = 2(1+x) = 2a$

$m'(a \otimes b) = m'((1+x) \otimes (x-1)) = x - 1 + 1 - x = 0 = m'(b \otimes a)$

$m'(b \otimes b) = m'((x-1) \otimes (x-1)) = 1 - x - x + 1 = 2(1-x) = -2b$

$\Delta'(a) = 1 \otimes x + x \otimes 1 + 1 \otimes 1 + x \otimes x = (1+x) \otimes (1+x) = a \otimes a$

$\Delta'(b) = x \otimes x + 1 \otimes 1 - (1 \otimes x + x \otimes 1) = b \otimes b$

Therefore, under the new basis

$m': V\otimes V \to V$ $\qquad\qquad$ $\Delta': V \to V\otimes V$

$a\otimes a \to 2a$ $\qquad\qquad$ $a \to a\otimes a$

$b\otimes b \to 2b$ $\qquad\qquad$ $b \to b\otimes b$

$a\otimes b \to 0$

$b\otimes a \to 0$

Example: left-handed trefoil

$d_{-1} \qquad\qquad d_0 \qquad\qquad d_1 \qquad\qquad d_2 \qquad\qquad d_3$

$$\begin{tikzcd}
 & V_1'\otimes V_2' \arrow[r] \arrow[dr] & V'' \arrow[dr] & \\
V_1\otimes V_2\otimes V_3 \arrow[ur] \arrow[r] \arrow[dr] & W_1'\otimes W_2' \arrow[ur] \arrow[dr] & W'' \arrow[r] & V_1'''\otimes V_2''' \\
 & X_1'\otimes X_2' \arrow[ur] \arrow[r] & X'' \arrow[ur] &
\end{tikzcd}$$

$d_0 = m'$

$a\otimes a\otimes a \longrightarrow 2(a\otimes a, a\otimes a, a\otimes a)$

$a\otimes a\otimes b \longrightarrow (a\otimes 0, 0\otimes a, 2a\otimes b)$

$a\otimes b\otimes a \longrightarrow (0, 2a\otimes b, 0)$

$a\otimes b\otimes b \longrightarrow (2a\otimes b, 0, 0)$

$b\otimes a\otimes a \longrightarrow (2b\otimes a, 0, 0)$

$b\otimes a\otimes b \longrightarrow (0, -2b\otimes a, 0)$

$b\otimes b\otimes a \longrightarrow (0, 0, -2b\otimes a)$

$b\otimes b\otimes b \longrightarrow -2(b\otimes b, b\otimes b, b\otimes b)$

Remaining basis elements: $(a\otimes a, 0, 0)$, $(0, a\otimes a, 0)$
$(b\otimes b, 0, 0)$, $(0, b\otimes b, 0)$

Since $\ker d_0 = 0$ then $\boxed{H_0 = 0}$
$\operatorname{Im} d_{-1} = 0$

$\underline{d_1 = m'}$

$(a\otimes a, 0, 0) \longrightarrow -2(a, a, 0)$

$(0, a\otimes a, 0) \longrightarrow -2(a, 0, a)$

$(b\otimes b, 0, 0) \longrightarrow 2(b, b, 0)$

$(0, b\otimes b, 0) \longrightarrow -2(b, 0, -b)$

Since $\ker d_1 = \operatorname{Im} d_0$ then $\boxed{H_1 = 0}$

$\underline{d_2 = \Delta'}$

$(0, 0, a) \longrightarrow a\otimes a$

$(0, b, 0) \longrightarrow -b\otimes b$

Since $\ker d_2 = \operatorname{Im} d_1$ then $\boxed{H_2 = 0}$

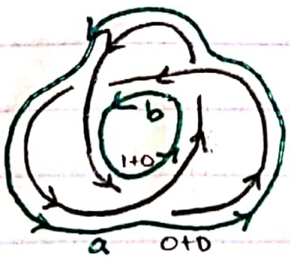
$\underline{d_3 = 0}$

$\ker d_3 = V\otimes V$ then $\boxed{H_3 = \mathbb{Q}\oplus\mathbb{Q} = \langle a\otimes b, b\otimes a\rangle}$

Thm: For any link L, $H_\bullet(L) = \bigoplus_{2^c} \mathbb{Q}$ where c = # of components of L

proof: 2^c = # orientations of link components

e.g. trefoil



orientation $\rightsquigarrow$ generator in Lee Homology

$\updownarrow$

oriented resolution

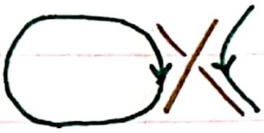
$\Rightarrow$ vertex in cube resolutions

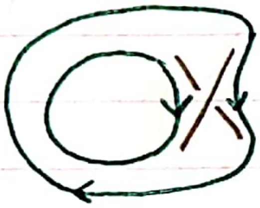
For a circle, choose a or b

= $\underbrace{\text{# circles containing this circle mod 2}}$ + $\begin{cases} 0, & \text{clockwise} \\ 1, & \text{counterclockwise} \end{cases}$

$\therefore$ for example: a-result = 0, b-result = 1

Key Lemma: If two circles share a crossing they are labeled differently

pf:  Case 1: Same # of circles containing but opposite orientations

 Case 2: Same orientation but different # of circles containing by 1 $\square$

Key Lemma: $\cdots \longrightarrow C_i \underset{d_i^*}{\overset{d_i}{\rightleftarrows}} C_{i+1} \underset{d_{i+1}^*}{\overset{d_{i+1}}{\rightleftarrows}} C_{i+2} \longrightarrow \cdots$

A chain complex. Assume C_i have nondegenerate bilinear forms on them

$H_i \cong \ker d_i \cap \ker d_{i-1}^* = \{x : d_i(x) = d_{i-1}^*(x) = 0\}$

NOTE: Relations to de Rham Cohomology & Hodge Theory

pf: $H_i = \dfrac{\ker d_i}{\operatorname{Im} d_{i-1}} = \ker d_i \cap (\operatorname{Im} d_{i-1})^{\perp} = \ker d_i \cap \ker d_{i-1}^*$ $\square$

For Khovonav / Lee Homology d^* goes backwards in the cube (m is dual to Δ) &

(m' is dual to Δ')