Tuesday, February 4, 2020

Chain complex

$$\cdots \to C_i \xrightarrow{d_i} C_{i+1} \xrightarrow{d_{i+1}} C_{i+2} \xrightarrow{d_{i+2}} \cdots$$

where C_i = vector spaces (or abelian groups)

with conditions $d^2 = d_{i+1} \circ d_i = 0 \quad \forall i$

$$H_i = \frac{\operatorname{Ker} d_i}{\operatorname{Im} d_{i+1}} \to \text{(co)homology}$$

NOTE: homology $\searrow$ & cohomology $\nearrow$

Recall: $A = \langle 1, x \rangle$ vector space over $\mathbb{Q}$
(or free abelian group of rank 2)

$m : A \otimes A \to A$

$1 \otimes 1 \mapsto 1$

$1 \otimes x \mapsto x$

$x \otimes 1 \mapsto x$

$x \otimes x \mapsto 0$

multiplication on A

$\mu : A \to A \otimes A$

$1 \mapsto 1 \otimes x + x \otimes 1$

$x \mapsto x \otimes x$

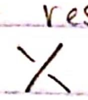
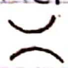
comultiplication

let $A = \mathbb{C}[x]/(x^2)$

Khovanov Homology

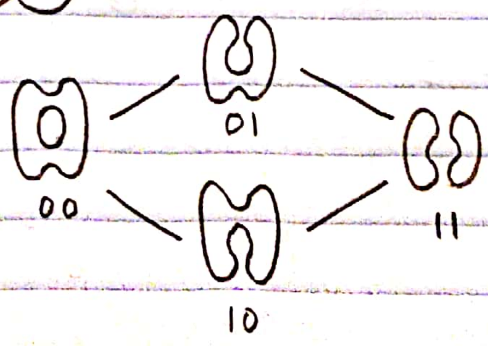
Input: link diagram

Step 0: Cube of resolutions

using  $\to$  (0-resolution)

$\to$  (1-resolution)

e.g. 



Step 1: At each vertex, place $A^{\otimes \#\text{circles}}$

e.g.

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, dash] & \\ A\otimes A \arrow[ur, dash] \arrow[dr, dash] & & A\otimes A \\ & A \arrow[ur, dash] & \end{tikzcd}$$

Step 2: On the edges, put m or μ

e.g.

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, "\mu"] & \\ A\otimes A \arrow[ur, "m"] \arrow[dr, "m"'] & & A\otimes A \\ & A \arrow[ur, "\mu"'] & \end{tikzcd}$$

Step 3: Add signs: $\text{sign}(\text{edge}) = (-1)^{\#1 \text{ before } i}$

(If switch 0 to 1 in i^{th} position)

$$\begin{tikzcd}[column sep=small] & 01 \arrow[dr, dash, "+"] & \\ 00 \arrow[ur, dash, "+"] \arrow[dr, dash, "+"'] & & 11 \\ & 10 \arrow[ur, dash, "-"'] & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, "\mu"] & \\ A\otimes A \arrow[ur, "m"] \arrow[dr, "m"'] & & A\otimes A \\ & A \arrow[ur, "-\mu"'] & \end{tikzcd}$$

NOTE: Every square has odd # of $(-)$

Step 4: $C_i = \bigoplus_{|v|=i}$ (spaces at vertex v)

d = sum of all edge maps w/signs

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, "\mu"] & \\ A\otimes A \arrow[ur, "m"] \arrow[dr, "m"'] \arrow[dd] & \oplus & A\otimes A \arrow[dd] \\ & A \arrow[ur, "-\mu"'] \arrow[d] & \\ C_0 & C_1 & C_2 \end{tikzcd}$$

Lemma: $d^2 = 0$

e.g. $m\mu + m(-\mu) = 0$

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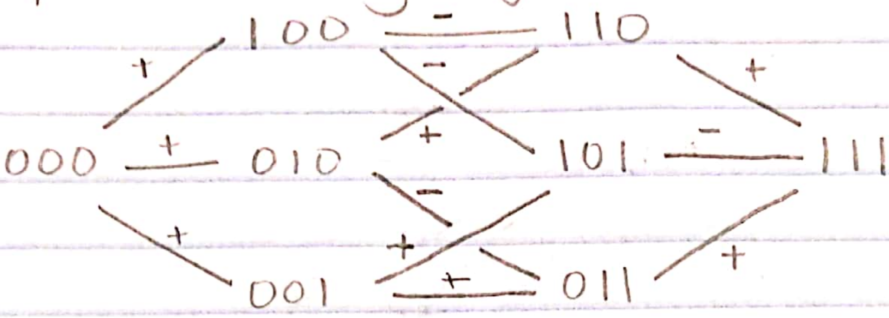
Example: Determining signs

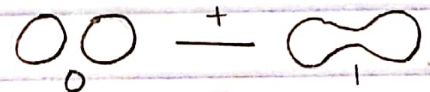


Thm: (Khovanov) $H_*(C_*, d)$ is a link invariant

Example:

1) Unknot: O A = Khovanov Homology of unknot

2) ∞



$$A\otimes A \xrightarrow{m} A$$

$1\otimes 1 \longrightarrow 1$

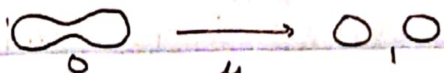
$1\otimes x \longrightarrow x \longleftarrow \text{Ker} = A = \text{Im}$

$x\otimes 1 \longrightarrow x \qquad \text{Ker}/\text{Im} = 0$

$x\otimes x$

$$\left.\begin{array}{l}\text{Ker} = x\otimes x \\ \quad\quad 1\otimes x - x\otimes 1 \\ \text{Im} = 0\end{array}\right\} H_0 = \text{Ker}$$

3) ∞



$$A \xrightarrow{\mu} A\otimes A$$

$1\otimes 1$

$1 \longrightarrow 1\otimes x$, $1 \longrightarrow x\otimes 1$

$x \longrightarrow x\otimes x$

Ker = 0

Im = 0

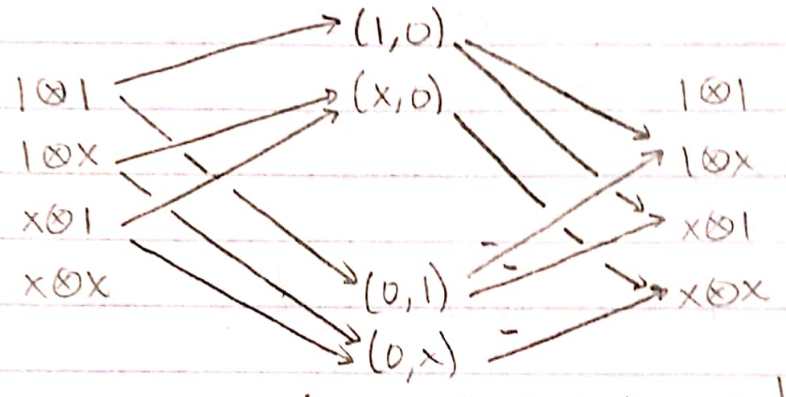
$\text{Ker} \subset A\otimes A$

$\text{Im} = \langle 1\otimes x + x\otimes 1, x\otimes x\rangle$

$H_* = \langle 1\otimes 1, 1\otimes x\rangle$

4)

$$\begin{tikzcd}[column sep=small] & A \arrow[dr] & \\ A\otimes A \arrow[ur] \arrow[dr] & & A\otimes A \\ & A \arrow[ur] & \end{tikzcd}$$



Ker $= x\otimes x$, $1\otimes x - x\otimes 1$	Ker $= (1,0)+(0,1) = (1,1)$, $(x,0)+(0,x)$ $=$ Im	Ker $= A\otimes A$
Im $= 0$		Im $= \left\langle \begin{matrix} x\otimes 1 + 1\otimes x \\ x\otimes x \end{matrix} \right\rangle$
$H = \left\langle \begin{matrix} x\otimes x \\ 1\otimes x - x\otimes 1 \end{matrix} \right\rangle$	$H = 0$	$H = \langle 1\otimes 1, 1\otimes x \rangle$
dim 2		dim 2

NOTE: 2 different homological degrees

HW: Compute Khovanov Homology for trefoil

<u>Two gradings</u>:

1) Homological $= |v|$ (# of 1's, up to overall shift)
   $\to$ want to change by 1
2) q-grading (up to overall shift)
   - A is graded $\deg_A(1) = 0$
     $\deg_A(x) = 2$ (convention)

   $\Rightarrow A^{\otimes k}$ is also graded
   - q-grading $= \deg_A - \#\text{circles} - |v|$

   $$\left.\begin{aligned} m:\ &\text{preserves } \deg_A \\ &\#\text{circles} \longrightarrow \#\text{circles} - 1 \\ &|v| \longrightarrow |v| + 1 \end{aligned}\right\} \Rightarrow \text{q-degree preserved}$$

$$\left.\begin{array}{l} \mu: \deg_A \longrightarrow \deg_A + 2 \\ \quad \#\text{circles} \longrightarrow \#\text{circles} + 1 \\ \quad |v| \longrightarrow |v| + 1 \end{array}\right\} \Rightarrow \text{q-grading preserved}$$

Conclusion:

d preserves q-grading & changes homological grading by 1.

Example:

1)  

$$\begin{array}{rlcll} -2 & 1\otimes 1 & \longrightarrow & 1 & -2 \\ 0 & 1\otimes x & \longrightarrow & x & 0 \\ 0 & x\otimes 1 & \nearrow & & \\ 2 & x\otimes x & & & \end{array}$$

2)  

$$\begin{array}{rlcll} -1 & 1 & & 1\otimes 1 & -3 \\ 1 & x & & 1\otimes x & -1 \\ & & & x\otimes 1 & -1 \\ & & & x\otimes x & 1 \end{array}$$

$1 \mapsto 1\otimes x,\ x\otimes 1$; $\quad x \mapsto x\otimes x$

preserves q-grading

$H^{i,j}$ = dim of homology in homological degree i, q-deg j

Claim: $\sum (-1)^i q^j \dim H^{i,j}$ = Jones polynomial

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