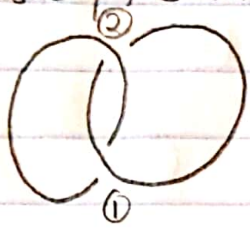
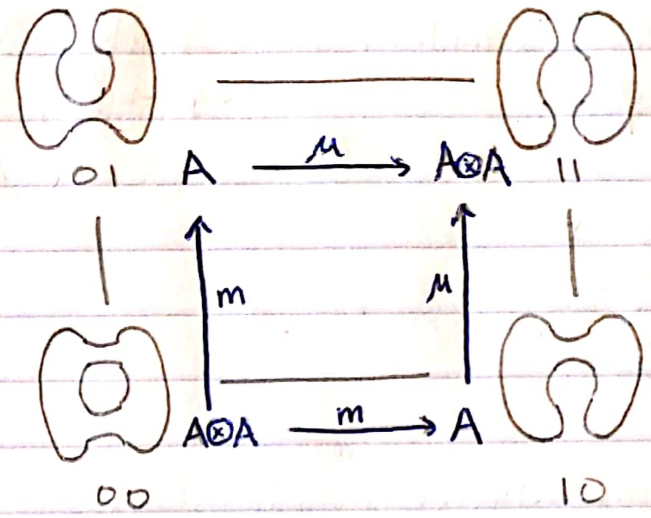
Tuesday, January 28, 2020



$$\times = \underbrace{)(}_{\text{0-res.}} - q \underbrace{\asymp}_{\text{1-res}}$$



vertices $\longrightarrow$ all possible resolutions

weight $\longrightarrow$ $(-q)^{|v|}(q+q^{-1})^{\#\text{circles}}$

where $|v| = \#$ of 1-resolutions

w/overall factor $(-1)^{\cdots} q^{\cdots}$

Categorification:

vertices $\rightsquigarrow$ vector spaces over $\mathbb{Q}$

edges $\rightsquigarrow$ linear maps

O circle $\rightsquigarrow$ A

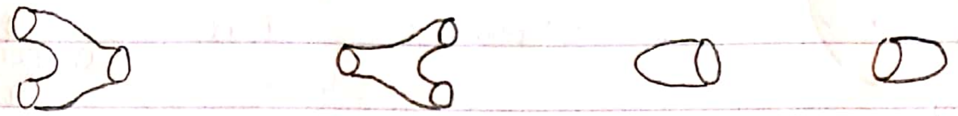
dim $A = 2$, graded $\leftsquigarrow$ $q + q^{-1}$

n circles $\rightsquigarrow$ $A^{\otimes n} = \underbrace{A \otimes \ldots \otimes A}_{n \text{ times}}$

$\dim A^{\otimes n} = 2^n \sim (q+q^{-1})^n$

Observation: On each edge, changing one crossing either merges two circles or splits one circle into two.

Q: What kind of structure do we need?



$A \otimes A \xrightarrow{m} A$ multiplication $x \otimes y \longrightarrow x \cdot y$

$A \xrightarrow{\mu} A \otimes A$ comultiplication

$Q \xrightarrow{i} A$ unit $1 \longrightarrow 1$

$A \xrightarrow{\varepsilon} Q$ counit

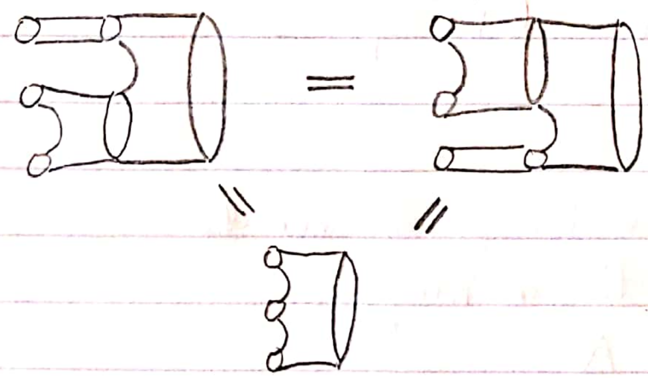
Example: $1 \cdot x = x$

NOTE: m is commutative & associative

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$A \otimes \underbrace{A \otimes A} \xrightarrow{m_{23}} A \otimes \underline{A} \xrightarrow{m} A$

$x \otimes y \otimes z \longrightarrow x \otimes (y \cdot z) \longrightarrow x \cdot (y \cdot z)$



NOTE:

$\therefore A$ is commutative, associative algebra

$A^* \longrightarrow A^* \otimes A^*$

Comultiplication (comultiplicative & coassociative)

Fact: A has a nondegenerate bilinear form

$(x, y) \longrightarrow \varepsilon(x \cdot y)$ so $A \simeq A^*$ & multiplication & comultiplication are dual to eachother

def: A w/all these structures & properties is called a <u>Frobenius algebra</u>
also called Topological Quantum F.T. of dim 1+1, aka TQFT where
1-dim mflds $\longrightarrow$ vector spaces
2-dim cobordisms $\longrightarrow$ maps

An important property for Frobenius algebra

(*)

$$\begin{tikzcd} A\otimes A \arrow[r, "\mu_2"] \arrow[dr, "m"'] & A\otimes A\otimes A \arrow[r, "\mu_{12}"] & A\otimes A \\ & A \arrow[ur, "\mu"'] & \end{tikzcd}$$

Example:
1) $A = \frac{\mathbb{Q}[x]}{(x^2)} = \langle 1, x\rangle$

Describe mult, comult, unit & counit

<u>Mult:</u>
$1\otimes 1 \to 1$
$1\otimes x \to x$
$x\otimes 1 \to x$
$x\otimes x \to 0$ (since $x^2=0$)

<u>Bilinear form:</u>
$(a,b) = \varepsilon(a\cdot b)$
$(1,1) = 0$
$(1,x) = (x,1) = 1$
$(x,x) = 0$

<u>Comult:</u>
$1 \longrightarrow 1\otimes x + x\otimes 1$
$x \longrightarrow x\otimes x$

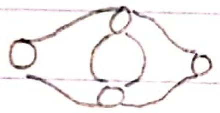
<u>Counit:</u>
$\varepsilon(1) = 0$
$\varepsilon(x) = 1$

Check $(*)$

$$\begin{tikzcd} 1\otimes 1 \arrow[r] \arrow[dr] & 1\otimes\underbrace{1\otimes x} + 1\otimes\underbrace{x\otimes 1} \arrow[r] & 1\otimes x + x\otimes 1 \\ & 1 \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} 1\otimes x \arrow[r] \arrow[dr] & 1\otimes\underbrace{x\otimes x} \arrow[r] & x\otimes x \\ & x \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} x\otimes x \arrow[r] \arrow[dr] & x\otimes x\otimes x \arrow[r] & 0 \\ & 0 \arrow[ur] & \end{tikzcd}$$



$$1 \longrightarrow 1\otimes x + x\otimes 1 \longrightarrow 2x$$

NOTE: There are <u>lots</u> of Frobenius algebras!
e.g. $\mathbb{Q}[x]/(x^n)$ or $\mathbb{Q}[x,y]/(x^n, y^m)$

Example:

1) $\mathbb{Q}[x]/(x^n) = \langle 1, x, \ldots, x^{n-1}\rangle$
where ε = coefficient at x^{n-1} & x^i dual to x^{n-1-i}

2) $\mathbb{Q}[x]/(x^2-1)$ Lee homology (later!)

Put m or Δ on edges of cube resolutions (pg 1 in blue)
Claim: If A is a Frobenius algebra then each 2d square commutes