Tuesday, March 3, 2020

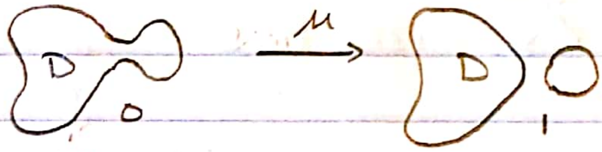
Thm: Khovanov Homology is a link invariant

→ Need to check Reidemeister moves

proof: R1|

(1a) (1b)

1a)



$$\begin{array}{lcl} C = C^{\cdot}(D) & \xrightarrow{\mu} & C^{\cdot}(D)\otimes A \\ \cup \quad\ \ \cup & & \quad \cup \\ C' \quad\ \ \emptyset & \xrightarrow{\mu} & C^{\cdot}(D)\otimes 1 \quad \text{subcomplex} \end{array}$$

$C/C' : C^{\cdot}(D) \xrightarrow{\mu} C^{\cdot}(D)\otimes A/\langle 1\rangle$ quotient

Claim: C/C' is acyclic, i.e. has no homology.

$\Rightarrow H_*(C) = H_*(C') \cong H_*(C^{\cdot}(D))$

proof:

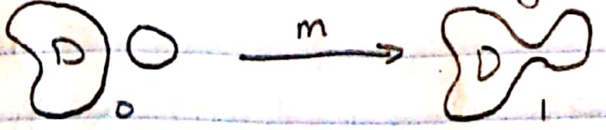
$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "\text{iso}"] & A\otimes A \arrow[r] & A\otimes A/\langle 1\rangle \end{tikzcd}$$

$$\begin{array}{lcl} 1 \longmapsto 1\otimes x + x\otimes 1 & \longmapsto & 1\otimes x \\ x \longmapsto x\otimes x & \longmapsto & x\otimes x \\ \therefore a \longmapsto & & a\otimes x \end{array}$$

Isomorphism; therefore, no homology

→ shift in homological degree □

1b)



$$\begin{array}{lcl} C^{\cdot}(D)\otimes A & \xrightarrow{m} & C^{\cdot}(D) = C \\ \quad \cup & & \\ C^{\cdot}(D)\otimes 1 & \xrightarrow{m} & C^{\cdot}(D) = C' \quad \text{subcomplex} \end{array}$$

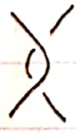
$C^{\bullet}(D) \otimes A/\langle 1 \rangle \longrightarrow \emptyset = C/C'$ quotient

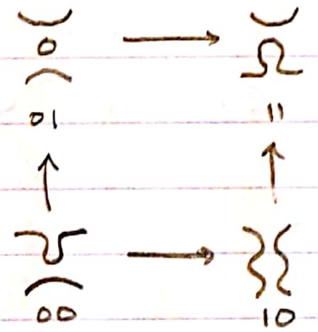
Claim: C' is acyclic

$\Rightarrow H_*(C) = H_*(C/C') \cong H_*(C^{\bullet}(D))$

up to grading shift

$\rightarrow$ shift in q degree $\square$

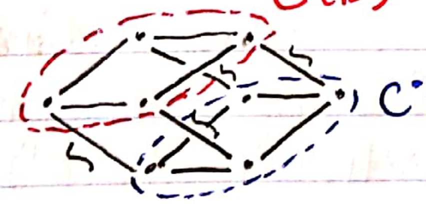
R2  Cube of resolutions:



Aside:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\cong"'] & \bullet \arrow[d, "\cong"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

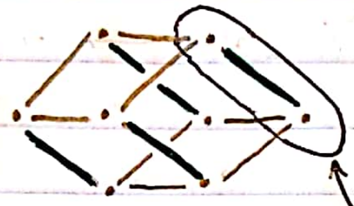
Claim: No homology



NOTE: $(A \otimes \dots \otimes \underbrace{A) \otimes A}_{\mu}$

(Snake) Lemma: For every $v \in \{0,1\}^n$ define K_v = some vector space, some differentials on edges which make it a chain complex. Suppose that for all $w \in \{0,1\}^{n-1}$ $K_{w_0} \xrightarrow{\sim} K_{w_1}$ is an isomorphism. Then cube is acyclic

Idea:



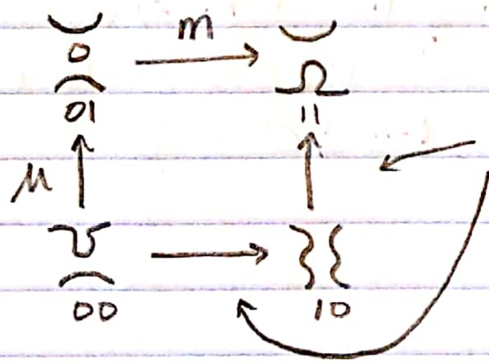
• all green edges are isomorphisms

acyclic complex

→ Take out acyclic subcomplexes one-by-one

quotient

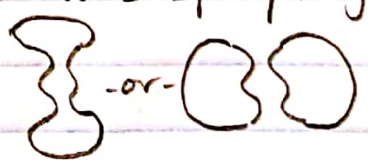
R2: cube of resolutions



Cannot determine maps

→ could be split/merge

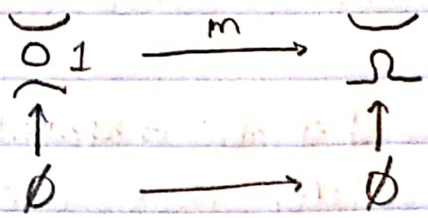
b/c



C

$\cup$

$C' =$



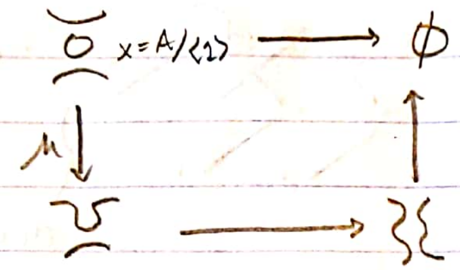
subcomplex

Claim: C' is acyclic

proof: b/c multiplication by 1 is an isomorphism

$A \otimes 1 \xrightarrow{m} A$ iso $\square$

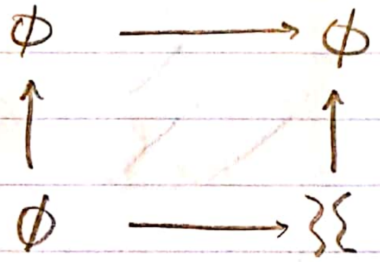
$C/C' =$



NOTE: each vertex represents $(n-2)$-cube

$\cup$

$C'' =$

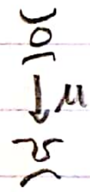


C'' is a subcomplex of C/C'

C' is acyclic, $(C/C')/C''$ is acyclic

$\therefore H_*(C) = H_*(C/C') = H_*(C'')$

where $(C/C')/C''$:



R3| Check!

Thm: Lee Homology is a link invariant

proof: Same proof as above

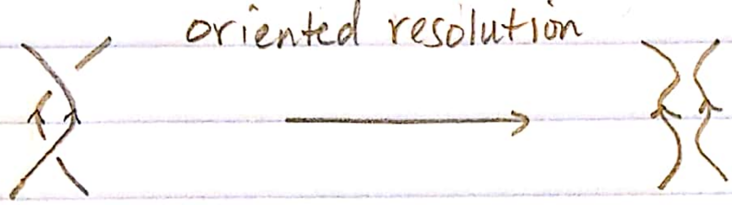
Fact: (Rasmussen) D, D' two diagrams related by a Reidemeister move (or a sequence of Reidemeister moves). Then there is a map

$$H_{Lee}(D) \xrightarrow{\varphi} H_{Lee}(D')$$

s.t. $\varphi\begin{pmatrix}\text{canonical}\\ \text{generator}\end{pmatrix} = \lambda \cdot \begin{matrix}\text{canonical}\\ \text{generator}\end{matrix}$, $\lambda \neq 0$

Recall: $H_{Lee}(D)$ has rank 2^{l}, if the link L has l components, $\mathcal{O}$ = orientation of L

$\rightsquigarrow$ $s_{\mathcal{O}}$ = canonical generator



→ In Lee & Rasmussen paper