Lee Spectral sequence

Tuesday, March 24, 2020 1:47 PM

Plan: 1) Recap on spectral sequences

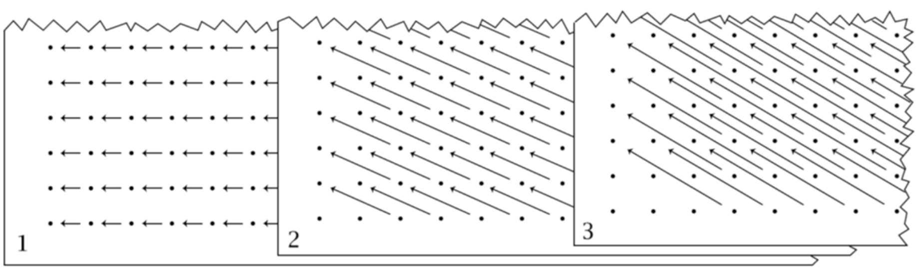
2) Recap on Serre spectral seq.

3) Example: Hopf fibration $S^3 \to S^2$

4) Recap: Khovanov and Lee differentials

5) Lee spectral sequence: construction

One can think of a spectral sequence as a book consisting of a sequence of pages, each of which is a two-dimensional array of abelian groups. On each page there are maps between the groups, and these maps form chain complexes. The homology groups of these chain complexes are precisely the groups which appear on the next page. For example, in the Serre spectral sequence for homology the first few pages have the form shown in the figure below, where each dot represents a group.



[Hatcher, Spectral sequences]

① Bicomplex

$$C_{i-1,j} \xleftarrow{d_1} C_{ij}$$

$$\begin{cases} d_0^2 = 0 \\ d_1^2 = 0 \end{cases} \qquad (*)$$

$$\begin{tikzcd} C_{i-1,j} & C_{ij} \arrow[l] \arrow[d, "d_0"] \\ & C_{i,j-1} \end{tikzcd} \qquad \begin{cases} d_1^2 = 0 \\ d_1 d_0 + d_0 d_1 = 0 \end{cases} \quad (*)$$

<u>Fact</u> $(*) \iff D^2 = 0$

where $D = d_0 + d_1$

$$\begin{tikzcd} d_1^2\ \bullet & \bullet \arrow[l, "d_1"'] \arrow[d, "d_0"'] & \bullet \arrow[l, "d_1"'] \arrow[d, "d_0"] \\ & d_1d_0+d_0d_1\ \bullet & \bullet \arrow[l, "d_1"] \arrow[d, "d_0"] \\ & & \bullet\ d_0^2 \end{tikzcd}$$

If $\{C_{ij}, d_0, d_1\}$ is a bicomplex

$\rightsquigarrow$ spectral sequence

Pages (E^r, d_r)

degree of $d_r = (-r, r-1)$

deg of $d_0 = (0, -1)$

deg of $d_1 = (-1, 0)$

degree of d_r is linear in r (

degree of d_r is linear in r!

d_r is a differential on page E^r

E^{r+1} = homology of d_r

In particular, $E^0 = \{C_{ij}\}$

$$E^1 = H_*(E^0, d_0) = H_*(C_{ij}, d_0)$$

$$E^2 = H_*(E^1, d_1), \ldots$$

Thm from last time: This spectral sequence converges to

$$E^\infty = \lim_{n\to\infty} E^n = H_*(C_{ij}, D)$$

total complex

Key example in topology: Leray-Serre spectral sequence

Thm $E \longrightarrow B$ fibration with fiber F, and $\pi_1(B) = 0$

— fiber F, and $\pi_1(B) = 0$

There is a spectral sequence

$$E^2 = H_*(B;\mathbb{K}) \otimes H_*(F;\mathbb{K})$$
$$\Downarrow \qquad\qquad \mathbb{K} = \text{field}$$
$$E^\infty = H_*(E)$$

Examples: ① Hopf fibration

$S^3 \longrightarrow S^2$ fiber S^1

$$E^2 = H_*(S^2;\mathbb{R}) \otimes H_*(S^1;\mathbb{R})$$

$$\begin{tikzcd}
S^1 & & & \\
1 & \mathbb{R} & & \mathbb{R} \\
0 & \mathbb{R} & & \mathbb{R} \arrow[llu, "d_2"'] \\
 & 0 & 1 & 2 \\
 & & & S^2
\end{tikzcd}$$

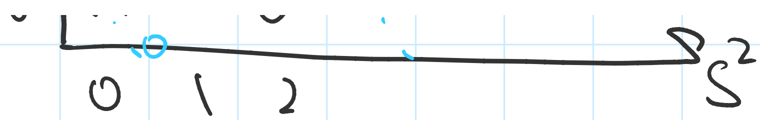
degree of $d_2 = (-2, 1)$

$$E^3 = H_*(E^2, d_2)$$

$$\begin{array}{c|ccc}
S^1 & & & \\
1 & 0 & & \mathbb{R} \\
0 & \mathbb{R} & & 0 \\
\hline
 & 0 & 1 & 2 \\
 & & & S^2
\end{array}$$

$d_3 = d_4 = \dots = 0$

3



$$E^3 = E^\infty = H_*(S^3)$$

degree = total degree $i+j$

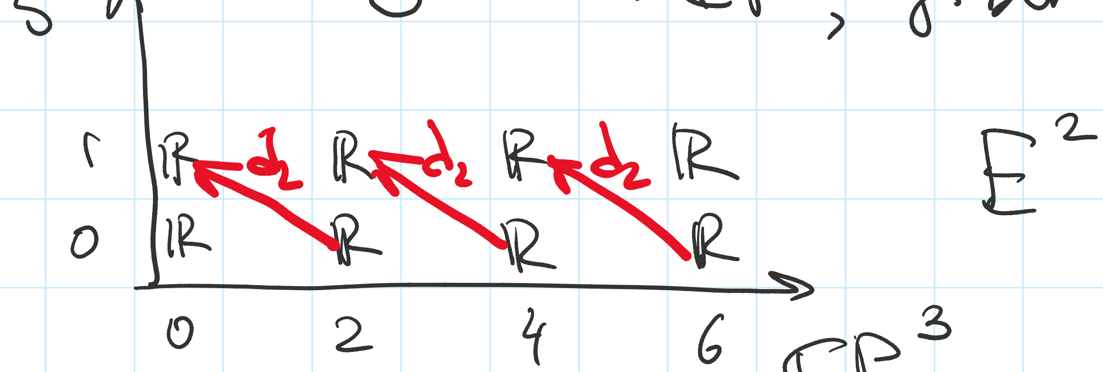
Exercise "Quaternionic Hopf fibration"

$S^7 \longrightarrow S^4$ fiber S^3

Ex: Compute the spectral sequence.

Example (from last time)

$S^{2n+1} \longrightarrow \mathbb{C}P^n$, fiber S^1



$$E^3 = H_*(E^2, d_2)$$
$$\shortparallel$$
$$E^\infty = H_*(S^7).$$

Back to Khovanov

Back to Khovanov

Khovanov	Lee
$m: 1\otimes 1 \to 1$	$m_{Lee}: 1\otimes 1 \to 1$
$1\otimes x \to x$	$1\otimes x \to x$
$x\otimes 1 \to x$	$x\otimes 1 \to x$
$x\otimes x \to 0$	$x\otimes x \to 1$
$\mu: 1 \to 1\otimes x + x\otimes 1$	$\mu_{Lee}: 1 \to 1\otimes x + x\otimes 1$
$x \to x\otimes x$	$x \to x\otimes x + 1\otimes 1$

$$m_{Lee} = m + \Phi_m$$

$$\mu_{Lee} = \mu + \Phi_\mu$$

Similarly, we can write

$$d_{Lee} = d_{Kh} + \Phi$$

<u>Observe</u>: $d_{Lee}^2 = 0 \quad d_{Kh}^2 = 0$

<u>Lemma</u> $\Phi^2 = 0$ and $d_{Kh}\Phi + \Phi d_{Kh} = 0$

so that d_{Kh} and Φ define a <u>bicomplex</u>.

[illegible] C_{Kh} [illegible]

Proof Recall we had q- and t-gradings on Kh complex

$\deg_A$ = grading on A

$\deg_A(1) = 0 \qquad \deg_A(x) = 2$

$\rightsquigarrow$ grading on $A \otimes A \otimes \dots \otimes A$

$$\deg_q = \deg_A - \#\text{circ} - |v|$$

for a resolution at vertex v

$|v| = \#$ ones in v

$$\deg_A(a \otimes b \otimes c) = \deg_A(a) + \deg_A(b) + \deg_A(c)$$

- m_{Kh} preserves $\deg_A$
- Φ_m shifts $\deg_A$ by -4
- μ_{Kh} shifts $\deg_A$ by $+2$
- Φ_μ shifts $\deg_A$ by -2

  $(\Phi_\mu(x) = 1 \otimes 1)$

- m_{Kh} shifts $\deg_q$ by $0 - \underset{\text{one less circle}}{(-1)} - (+1)$

- m_{Kh} shifts $\deg_q$ by $0-(-1)-(+1)$
$= 0$
- Φ_m shifts $\deg_q$ by $-4-(-1)-(+1)$
$= -4$
- μ_{Kh} shifts $\deg_q$ by $+2-(+1)-(+1)$
one more circle $= 0$
- Φ_μ shifts $\deg_q$ by $-2-(+1)-(+1)$
$= -4$

Both m, μ preserve q-grading
Both Φ_m, Φ_μ shift q-grading by -4

<u>Conclusion</u> d_{Kh} has bidegree $(q,t) = (0,1)$
Φ has bidegree $(-4, 1)$

$(d_{Kh} + \Phi)^2 = 0 \implies$

$$\underbrace{\Phi^2 = 0}_{(-8,2)} \text{ and } \overbrace{d_{Kh}\Phi + \Phi d_{Kh}}^{(-4,2)} = 0$$

by Lemma. □

<u>Thm</u> (Lee, Rasmussen) There is a spectral sequence with

spectral sequence with

E^0 = Khovanov complex, $d_0 = d_{Kh}$

$E^1 = H_*(E^0, d_0) =$ Khovanov homology of K

$d_1 = \Phi$, $E^2 = H_*(E^1, \Phi)$

$E^\infty = H_*(E^0, d_0 + \Phi) =$

$= H_*(E^0, d_{Lee}) =$ Lee homology of K

$\boxed{\text{degree of } d_r = (-4r, 1)}$

Example: trefoil, we computed

dim (Khovanov homology) = dim (E_2)

$= 4$

dim (Lee homology = dim (E_∞)

$= 2$

$\Rightarrow$ there are interesting differentials.

More details next time!