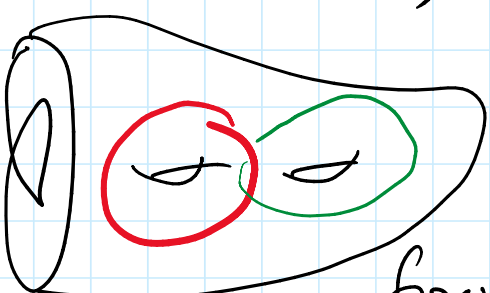
Seifert form and Alexander polynomial

Thursday, May 21, 2020 2:01 PM

K = knot in S^3

Σ = a Seifert surface for K of genus g

(oriented smooth surface in S^3, $\partial \Sigma = K$).



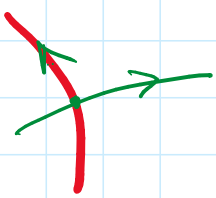
$H_1(\Sigma, \mathbb{Z}) = \mathbb{Z}^{2g}$

Intersection form

$$(\cdot,\cdot) : H_1(\Sigma,\mathbb{Z}) \otimes H_1(\Sigma,\mathbb{Z}) \to \mathbb{Z}$$

skew-symmetric, nondegenerate

$$(x,y) = -(y,x)$$



Seifert form

$$S(x,y) = lk(x^+, y)$$ ← not symmetric or antisymmetric

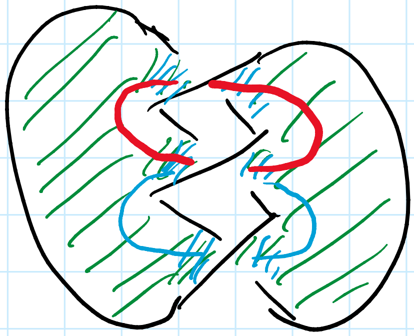
where x^+ = pushoff of x in positive normal direction to Σ

Choose some basis of $H_1(\Sigma, \mathbb{Z})$

Choose some basis on $H_1(\Sigma, \mathbb{Z})$

$\rightsquigarrow$ represent $\mathcal{S}$ by some matrix S.

<u>Ex</u>



trefoil

Σ = two disks connected by twisted bands at crossings.

genus = 1 $\quad H_1 = \mathbb{Z}^2$

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

antisymmetric

<u>Fact</u>: $\mathcal{S} - \mathcal{S}^T$ = intersection form

$$\mathcal{S}(x,y) - \mathcal{S}(y,x) = (x,y)$$

$$lk(x^+,y) - lk(x,y^+)$$

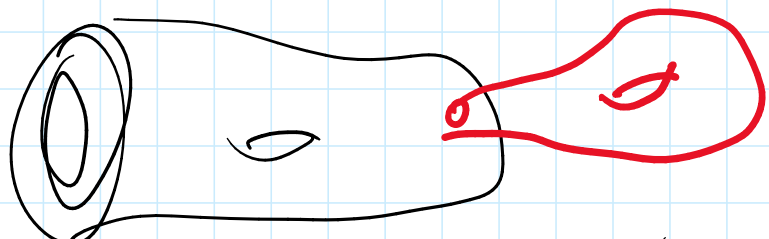
$$S - S^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

<u>Def</u> The <u>Alexander polynomial</u> of $K = \det(S - tS^T) = \Delta_K(t)$

The <u>signature</u> of K is the signature $\sigma(K)$ of the symmetric bilinear form defined of the symmetric bilinear form defined by the matrix $S+S^T$ $S(x,y)+S(y,x)$

Fact $\Delta_K(t)$ and $\sigma(K)$ are invariants of K and do not depend on the choice of Seifert surface.

Idea Two different Seifert surfaces for the same knot K are related by a sequence of stabilizations and destabilizations



One can check that under stabilization the Seifert matrix changes as

$$\tilde{S} = \begin{pmatrix} \boxed{S} & * \\ & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

then one can check that S and $\tilde{S}$ yield the same Alexander polynomial (up to multiplication by t^i) and signature (up to multiplication by $t^{\cdot}$) and signature.

Indeed. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ sign $= 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix} \quad \det = t$$

Normalized Alexander polynomial: $\tilde{\Delta}(t) = t^{\cdots} \Delta(t)$

$$\tilde{\Delta}(t^{-1}) = \Delta(t).$$

Indeed, $\det(S - tS^T) \overset{\text{transpose}}{=} \det(S^T - tS)$

$$\det(S - t^{-1}S^T) = t^{-2g} \det(tS - S^T)$$

Ex For trefoil $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\Delta(t) = \det(S - tS^T) =$$

$$= \det \begin{pmatrix} 1-t & 1 \\ -t & 1-t \end{pmatrix} = (1-t)^2 + t = 1 - 2t + t^2 + t$$

$$= 1 - t + t^2$$

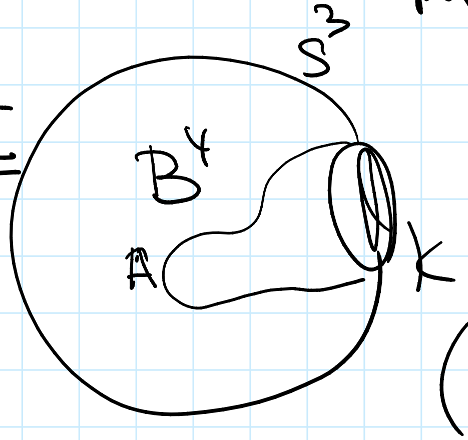
Normalize: $(1 - t + t^2) \cdot t^{-1} = t^{-1} - 1 + t$

$\sigma =$ signature of $S + S^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 2$

both eigenvalues are positive.

both eigenvalues are positive.

Fact

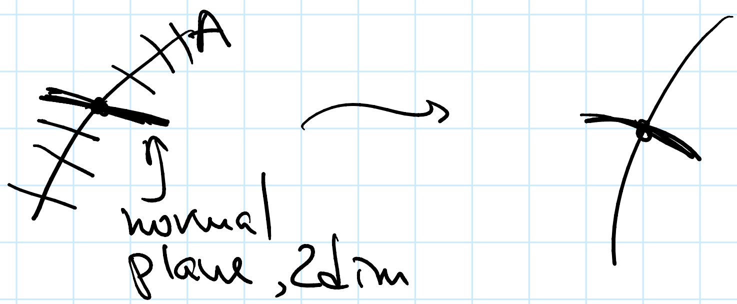


A = some surface in B^4 with boundary K (for example, it could be Seifert surface pushed to B^4)

M = double cover of B^4 branched along A

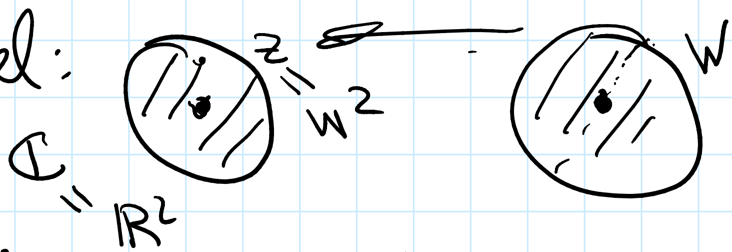
i.e.: away from A = double cover

On A:



do this in normal direction at every point of A

local model:



If $z = w^2$, every point except 0 has 2 preimages $\to$ double cover outside 0

Remark Branched double cover is

Rmk Branched double cover is defined for a pair $A^m \subset B^{m+2}$, A is a submanifold in B of codimension 2

M^4 = branched double cover of B^4 branched along A

∂M^4 = branched double cover of S^3 along K

$H_2(M^4)$ has intersection form

$H_2(M^4) \times H_2(M^4) \longrightarrow \mathbb{Z}$ symmetric

Fact The signature of the intersection form on $H_2(M^4)$ equals $\sigma(K)$ and does not depend on the choice of surface A.

Why do we care about $\Delta_K(t)$, $\sigma(K)$?

Facts: (1) Seifert genus of K is bounded by the Alexander polynomial!

$\Delta(t)$ = normalized Alexander poly

$\tilde{\Delta}(t)$ = normalized Alexander poly

$$\boxed{-g_3(K) \le \deg \tilde{\Delta}(t) \le g_3(K)}$$

<u>Proof</u>: $\Delta(t) = \det(S - tS^T)$

$\qquad\qquad\qquad\qquad$ ↖ $2g \times 2g$ matrix

$\Rightarrow \Delta(t)$ is a polynomial of degree at most $2g$.

$0 \le \deg \Delta(t) \le 2g$ ← for any choice of Seifert surface of genus g

$-g \le \deg \tilde{\Delta}(t) \le g$

Seifert genus $g_3(K) = \min_{\text{Seifert surface}} \{g\}$.

<u>Ex</u> For trefoil $\tilde{\Delta}(t) = t^{-1} - 1 + t$

$\Rightarrow g_3(\text{Trefoil}) \ge \deg \tilde{\Delta}(t) = 1$

② [Murasugi] $\boxed{2g_4(K) \ge |\sigma(K)|}$

g_4: 4-genus (slice genus).

<u>Ex</u> For trefoil $\sigma(K) = 2$

$2g_4(K) \ge 2 \qquad g_4(K) \ge 1$

Since we have a S. surface of genus 1

Since we have a 2-surface of genus 1

$$g_3(\text{trefoil}) = g_4(\text{trefoil}) = 1.$$

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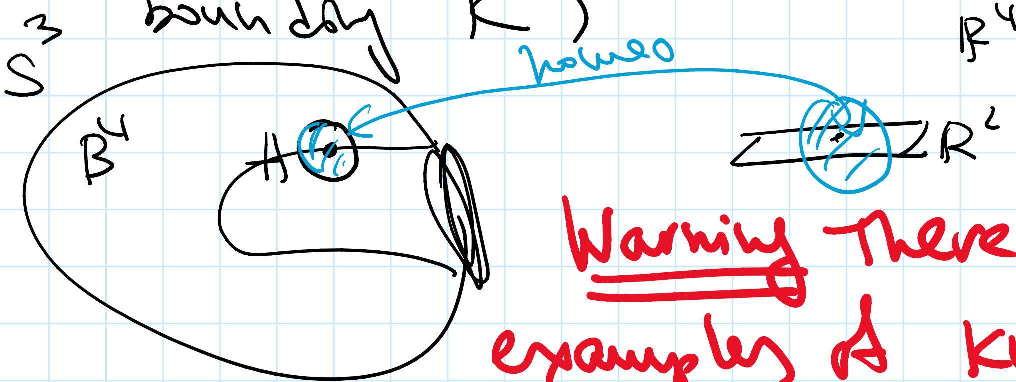
③ [Freedman, hard!] If $\tilde{\Delta}_K(t) = 1$ then K is topologically slice ($\exists$ top. locally flat disk A in B^4 boundary K)

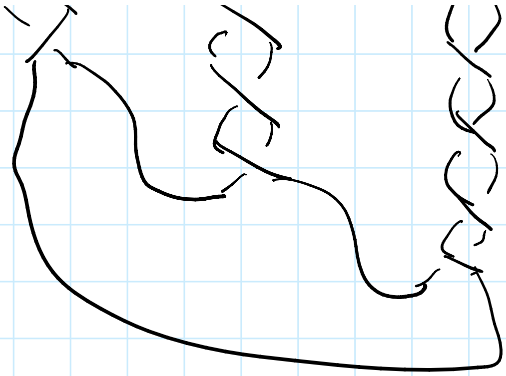


Warning There are examples of knots which are topologically slice and not smoothly slice. As Lisa mentioned, they can be used to construct exotic smooth structures on $\mathbb{R}^4$.

Ex There are knots (ex. pretzel knot $P(-3,5,7)$)

Can check:

$\Delta = 1 \Rightarrow$ by



$\Delta = 1 \Rightarrow$ by Freedman's thm it is top. slice

Rasmussen: $s = -1$

$\Rightarrow$ not smoothly slice!

(4) [Fox - Milnor] If K is smoothly slice then $\Delta_K(t) = f(t) f(t^{-1})$ where f is some polynomial with integer coefficients.

Ex $K \# \overline{K}$ is smoothly slice

$$\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t)\, \Delta_{K_2}(t)$$

$$\Delta_{K \# \overline{K}}(t) = \Delta_K(t)\, \Delta_K(t^{-1}).$$

If $\Delta_K(t)$ is irreducible (or not of the form $f(t) f(t^{-1})$) then K is not smoothly slice.

If $\sigma(K) \neq 0$ then K is neither top. nor smoothly slice by Murasugi's theorem knot smoothly slice by Murasugi's theorem,