Properties of s-invariant

Thursday, April 30, 2020 2:11 PM

Properties of s-invariant

Recall: $s = \frac{s_{min} + s_{max}}{2}$

s_{min}, s_{max} = min/max q-degrees of generators in Lee homology.

① $\overline{K}$ = mirror of K, then

$$s_{max}(\overline{K}) = -s_{min}(K)$$
$$s_{min}(\overline{K}) = -s_{max}(K)$$

so $s(\overline{K}) = -s(K)$.

<u>Proof</u>: Both Khovanov and Lee complexes for K and $\overline{K}$ are dual to each other.

More precisely: A has a bilinear form

$$\langle 1, x \rangle = (x, 1) = 1$$
$$(1,1) = (x,x) = 0$$

this identifies $A \simeq A^*$

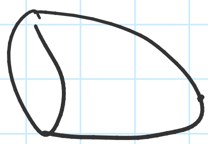
Under this identification, μ is dual to Δ

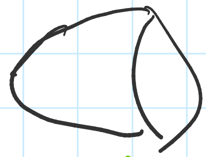
Under this identification, μ is dual to m

$\varepsilon : A \to \mathbb{C}$ $\qquad$ $i : \mathbb{C} \longrightarrow A$

$\varepsilon(x) = 1$ $\qquad$ $i(1) = 1$

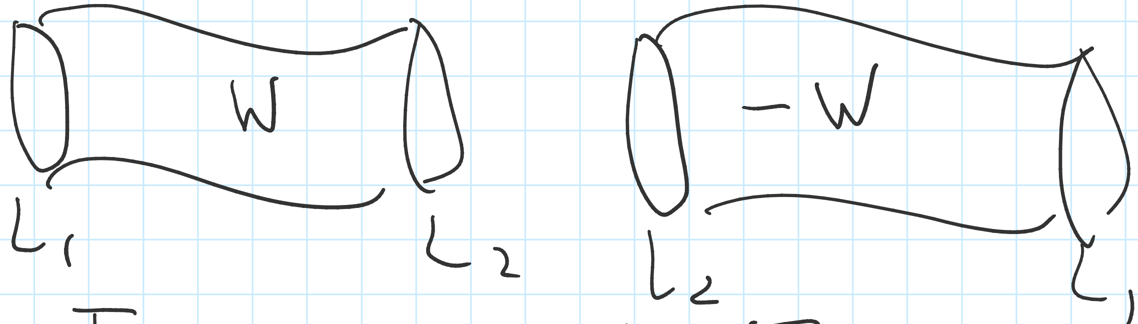
$\varepsilon(1) = 0$





also dual to each other

Cor W = cobordism from L_1 to L_2



Φ_{-W} is dual to Φ_W.

0-resolutions for K $\longleftrightarrow$ 1-resolutions for $\overline{K}$

$\Rightarrow$ cube of resolutions is reflected

$C_{Kh}(K)$ is dual to $C_{Kh}(\overline{K})$

$\Rightarrow$ their homology (with coefficients in a field) are dual a field) are dual to each other.

$\Rightarrow$ generators of Lee homology of K correspond to generators of Lee homology of $\overline{K}$ with reversed q-degrees. $\square$

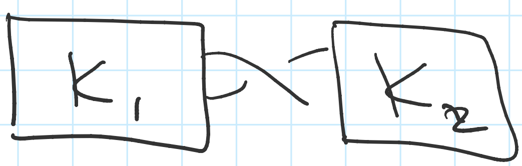
Cor We know $|s(K)| = 2g_*(K)$ for positive K

$\overline{K}$ is negative $\Rightarrow$ $|s(\overline{K})| = |s(K)| = 2g_*(K)$

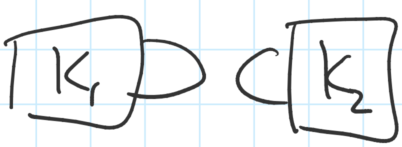
Fact (Rasmussen) $s(K_1 \# K_2) = s(K_1) + s(K_2)$

($\#$ = connect sum)

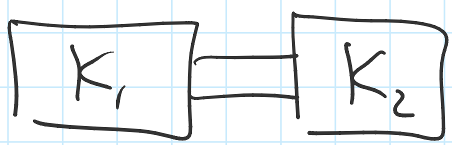
Idea: use



$K_1 \# K_2$



$K_1 \sqcup K_2$



also $K_1 \# K_2$

Cor $s(K \# \overline{K}) = s(K) + s(\overline{K}) =$

$= s(K) - s(K) = 0$

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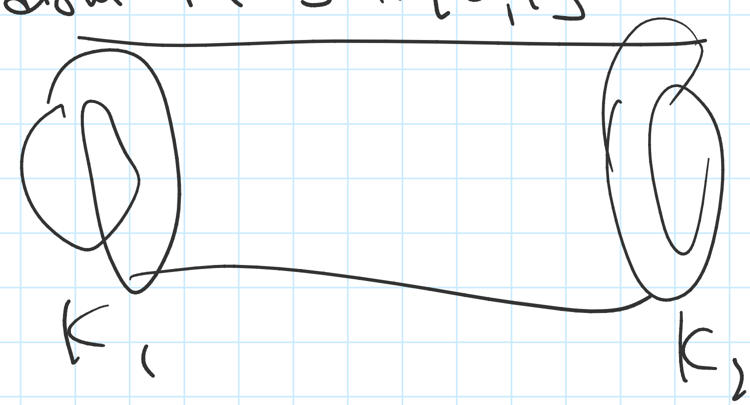
This is consistent with genus bound:
as James discussed last time, $K \# \overline{K}$ is slice

$g_*(K \# \overline{K}) = 0 \geq |s(K \# \overline{K})|$

Def K_1 and K_2 are <u>concordant</u> if they are connected by a genus 0 cobordism in $S^3 \times [0,1]$



$\Updownarrow$

$K_1 \# \overline{K_2}$ is slice

<u>Concordance group</u> $\mathcal{C}$: generators = knots / concordance relation, operation = #

identity = unknot $O \# K = K$

$-K = \overline{K}$ inverse

$K \# \overline{K}$ is slice
$\Leftrightarrow K \# \overline{K}$ concordant to unknot.

Thm (Rasmussen) s-invariant defines a

Thm (Rasmussen) s-invariant defines a homomorphism from $\mathcal{C}$ to $\mathbb{Z}$.

Proof: Need to check:

(a) $s(K_1 \# K_2) = s(K_1) + s(K_2)$ ✓

(b) If K_1 and K_2 are concordant then $s(K_1) = s(K_2)$

(b): If K_1 and K_2 are concordant then $K_1 \# \overline{K_2}$ is slice $\Rightarrow |s(K_1 \# \overline{K_2})| \leq g_*(\dots) = 0$

$$\Rightarrow s(K_1 \# \overline{K_2}) = 0$$

$$s(K_1 \# \overline{K_2}) = s(K_1) + s(\overline{K_2}) = s(K_1) - s(K_2)$$

therefore $s(K_1) = s(K_2)$.

Ex T = trefoil

$s(T) = -2$ in our conventions.

$s(\overline{T}) = 2$

$s(T \# T \# \dots \# T) = -2K$

$s(\overline{T} \# \overline{T} \# \dots \# \overline{T}) = 2K$

$s(T \# T \# \dots \# T) = 2k$

$\Rightarrow$ any even integer appears as a value of s-invariant.

<u>Thm</u> [Hom, ...] $\mathcal{C}$ has a $\mathbb{Z}^\infty$ direct summand

(Proof uses Heegaard-Floer homology)

$\mathcal{C}$ is a very complicated group!

Plan for the following weeks:

Understand why Conway knot is not slice.

Why interesting:

Conway knot is related to another knot called Kinoshita-Terasaka knot by <u>mutation</u>

Many invariants do not change under mutation $\Rightarrow$ hard to distinguish

KT knot is slice

But Conway knot is not slice

K , but is slice

But Conway knot is not slice.

Strategy of pf:

① $K \rightsquigarrow X(K)$
knot trace

K is slice $\Leftrightarrow$
$X(K)$ embeds in S^4

② Sometimes $X(K) = X(K')$
for different knots K, K' [4d top. argument]

③ In particular, for K = Conway knot
one can find K' as above such that
$s(K') \neq 0$

Then $g_*(K') \geq |s(K')| > 0$

$\Rightarrow K'$ is not slice $\Rightarrow X(K') = X(K)$
does not embed in S^4

$\Rightarrow$ K is not slice. □.

$s(\text{Conway}) = s(KT) = 0$
$\Rightarrow$ cannot use Rasmussen's genus bound