Meeting 1/21

Thursday, January 21, 2021 2:51 PM

1. (Riemann-Hurwitz formula) Let C be a smooth surface (maybe with boundary) and $\widetilde{C} \to C$ a branched cover of degree d. Every point of C has d preimages except for n branch points which have $k_1, \ldots, k_n$ preimages respectively.

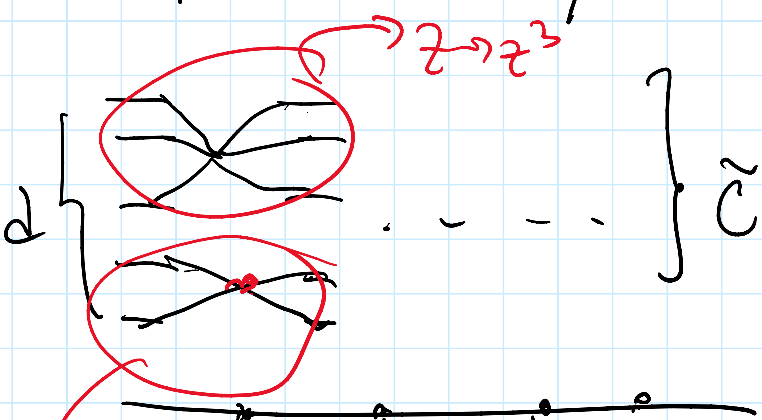
(a) Prove that

$$\chi(\widetilde{C}) = d\chi(C) + \sum_{i=1}^{n}(k_i - d)$$

(b) Assume that C and $\widetilde{C}$ have no boundary. Find a relation between the genus of C and the genus of $\widetilde{C}$.

$\tilde{C} \xrightarrow[\text{n-fold cover}]{} C$

$\chi(\tilde{C}) = n \cdot \chi(C)$ [every symplex in C is covered by n simplices of the same dimension]



$C \leftarrow$ surface

d-fold cover except for special branch point.

n branch points on C

$f: \mathbb{C} \to \mathbb{C}$

$z \to z^2$

$f^{-1}(w) = 2$ points for $w \neq 0$

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1 point for $w = 0$

- $\chi(C - n\,\text{pts}) = \chi(C) - n$
- $\chi(\tilde{C} - \text{branch points}) = d \cdot \chi(C - n\,\text{pts})$

$$= d(\chi(C) - n)$$

- $\chi(\tilde{C}) = d(\chi(C) - n) + \Sigma k_i$

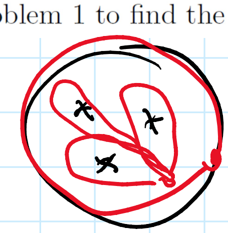
$$= d\chi(C) + \Sigma(k_i - d)$$

2. (a) Prove that $\{y^2 = x^3 + \varepsilon\}$ is a smooth algebraic curve in $\mathbb{C}^2$ for $\varepsilon \neq 0$.
(b) Use Problem 1 to find its genus by projecting to one of the axis.

$\frac{\partial f}{\partial x} = 3x^2 \quad \frac{\partial f}{\partial y} = 2y$

3. (a) Prove that the link of singularity $\{x^m = y^n\}$ is the (m, n) torus link with $GCD(m, n)$ components.
(b) Prove that $\{y^m = x^n + \varepsilon\}$ is a smooth algebraic curve in $\mathbb{C}^2$ for $\varepsilon \neq 0$.
(c) Use part (a) and Problem 1 to find the genus of this curve.

sing $(0,0)$ not on the curve.



3 branch pts

$y = \pm\sqrt{x^3 + \varepsilon}$

normally 2 preimages.

$\chi(\mathbb{C} - 3\,\text{pts}) = 1 - 3 = -2$

$\shortparallel$

base − branch pt.

base - branch pts.

χ(curve - branch pts) $= 2\cdot(-2) = -4$

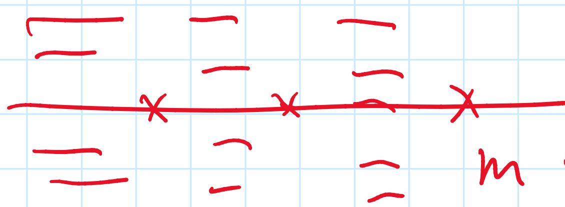
χ(curve) $= -4 + 3 = -1$

Note: 1 bdry component, connected

$(2-2g) - \underset{\text{puncture}}{1} = -1$

$2-2g = 0$ $\quad g=1$ $\quad$ torus w 1 puncture

$x^m = y^n + \varepsilon \qquad y = \sqrt[n]{x^m - \varepsilon}$



m branch pts.

1 preimage

every other pt = n preim.

$\chi = n(1-m) + m$

$\quad = n + m - mn$

Monodromy around big circle =

$= (1 \dots n)^m \in S_n$

$= (1 \dots n)^m \in S_n$

has d components $d = GCD(m, n)$