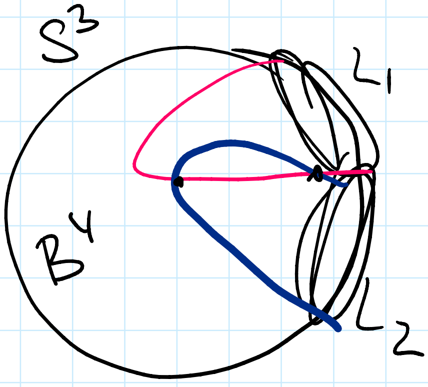
Meeting 2/11

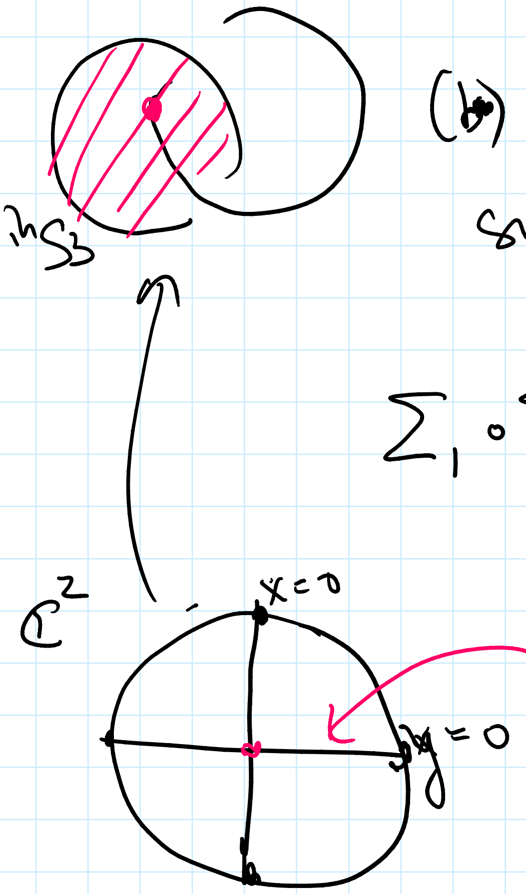
Thursday, February 11, 2021 2:57 PM



$lk(L_1, L_2) = ?$

(a) find a surface Σ_1 in S^3 such that $\partial\Sigma_1 = L_1$

then $lk(L_1, L_2) = \Sigma_1 \circ L_2$ ← intersection #



(b) find a pair of surfaces Σ_1, Σ_2 in B^4

$\partial\Sigma_i = L_i$

$\Sigma_1 \circ \Sigma_2$ = intersection # in B_4

$= lk(L_1, L_2)$

Σ_1, Σ_2 = two discs in B^4

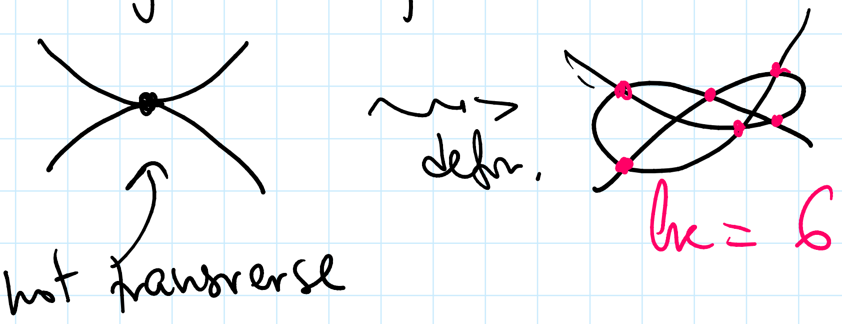
$\partial(xy=0)$ = Hopf link

If L_1, L_2 are algebraic links, we can do (b) algebraically by choosing Σ_1, Σ_2 to be the corresponding Milnor fibers the corresponding Milnor fibers

$\Sigma_1 = \{\tilde{f} = 0\}$ $\Sigma_2 = \{\hat{g} = 0\}$

$\Sigma_1 \cdot \Sigma_2$ = algebraic intersection number of $\{\tilde{f} = 0\}$ $\{\hat{g} = 0\}$.

Ex: $f = \{x^2 - y^3\}$
$g = \{x^2 + y^3\}$



Algebraic fact: One can compute the intersection number as follows:

- Parametrize $\{f = 0\}$ by $(x(t), y(t))$ (provided that it's irreducible)
- Compute the order of $g(x(t), y(t))$

$\{f = 0\} \longrightarrow$ parametrize (t^3, t^2)

$g = x^2 + y^3 \Rightarrow g(x(t), y(t)) =$
$= t^6 + t^6 = 2t^{\textcircled{6}}$

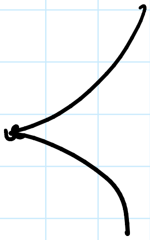
lk = 6.

$= t^6 + t^6 = 2t^{\boxed{6}}$

$lk = 6.$

Note: Also $lk = \dim \dfrac{\mathbb{C}[[x,y]]}{(f,g)}$

Semigroup of plane curve singularity



$(x(t), y(t)) =$ parametrization

$S = \{ \operatorname{Ord} g(x(t), y(t))$ for all possible $g(x,y) \}$,

(= all possible linking #'s with algebraic links).

Ex: $(\underset{x}{t^2}, \underset{y}{t^3})$ $S = \{\underset{1}{0}, \underset{x}{\overset{\mu}{\boxed{2}}}, \underset{y}{3}, \underset{x^2}{4}, \underset{xy}{5}, \dots\}$

no function of order 1

Ex $\underset{\langle 3,5 \rangle}{}(\underset{x}{t^3}, \underset{y}{t^5})$ $S = \{\underset{1}{0}, \underset{x}{3}, \underset{y}{5}, \underset{x^2}{6}, \underset{xy}{\overset{\mu}{\boxed{8}}}, \underset{x^3}{9}, \underset{y^2}{10}, \dots\}$

Have coins valued 3 and 5 cents, which amounts can we change with these coins?

$S = \{3a + 5b \mid a, b \geq 0\}$

$$S=\{3\underset{x^a y^b}{a+5}b \mid a,b\geq 0\}$$

Ex $(x=t^4, y=t^6+t^7)$

$$S=\{\underset{\overset{\uparrow}{1}}{0}, \underset{\overset{\uparrow}{x}}{4}, \underset{\overset{\uparrow}{y}}{6}, \underset{\overset{\uparrow}{x^2}}{8}, \underset{\overset{\uparrow}{xy}}{10}, \underset{\overset{\uparrow}{y^2}}{12}, 13, \underset{\overset{\uparrow}{x^2y}}{14}, \quad , \underset{\overset{\uparrow}{x^4}}{\overset{\mu}{\boxed{16}}}, \underset{\overset{\uparrow}{13+4}}{17}, \underset{\overset{\uparrow}{3\cdot 6}}{18}, \underset{\overset{\uparrow}{13+6}}{19} \ldots$$

$$\| \\ \langle 4,6,13\rangle$$

$$y^2-x^3=(t^6+t^7)^2-t^{12}$$
$$=t^{12}+2t^{13}+t^{14}-t^{12}=$$
$$=2\underline{\underline{t^{13}}}+t^{14}$$

Facts: 1) (Semigroup) $\underset{\text{ord}\, g_1}{\underset{\|}{s_1}}, \underset{\text{ord}\, g_2}{\underset{\|}{s_2}}\in S \Rightarrow \underset{\text{ord}(g_1g_2)}{\underset{\|}{s_1+s_2}}\in S$

2) (Conductor) For all $s\geq \mu \leftarrow$ μ: Milnor number
we have $s\in S$

3) (Symmetry) $x\in S \Leftrightarrow \mu-1-x\notin S$

4) Semigroup is determined and determines Puiseaux pairs ($\approx$ generators of S)

If there are p Puiseaux pairs

of S)

If there are p Puiseaux pairs, S has $p+1$ generators.

Thm (Campillo, Delgado, Gusein-Zade)

$$\sum_{s\in S} q^s = \frac{\Delta(q)}{1-q}$$

Alexander polynomial of the link!

Ex (t^2,t^3) $S=\{0,2,3,\dots\}$

$$1+q^2+q^3+\dots = 1+\frac{q^2}{1-q} = \frac{1-q+q^2}{1-q}$$

Alexander poly of $T(2,3)$

Ex (t^3,t^5) $S=\{0,3,5,6,8,\dots\}$

$$1+q^3+q^5+q^6+q^8+\dots \underset{\text{exercise}}{=} \frac{(1-q^{15})}{(1-q^3)(1-q^5)} = \frac{\Delta(q)}{1-q}$$

Ex (t^4,t^6+t^7)

$S=\{0,4,6,8,10,12,13,14,16,\dots\}$

$$1+q^4+q^6+q^8+q^{10}+q^{12}+q^{13}+q^{14}+q^{16}+\dots$$

$$1+q^4+q^6+q^8+q^{10}+q^{12}+q^{13}+q^{14}+q^{16}+\dots$$

$$= \frac{(1-q^{12})(1-q^{26})}{\underbrace{(1-q^4)(1-q^6)(1-q^{13})}_{\text{generators of } S}}$$

Symmetry of S $\longleftarrow$ $\Delta(q^{-1}) = \Delta(q)\,q^{\dots}$

<u>Note:</u> K = any knot   cabling formula.

$\tilde{K}$ = cable of K

$$\Delta_{\tilde{K}} = \Delta_K(q^{\dots}) \cdot \frac{(1-q^{\dots})}{(1-q^{\dots})}$$

<u>Ex</u>: $T(2,3) \longrightarrow \Delta = \frac{1-q+q^2}{1-q} = \boxed{\frac{(1-q^6)}{(1-q^2)(1-q^3)}}$

change $q \to q^2$

Link of (t^4, t^6+t^7)

$= (\underline{2},\underline{13})$ cable of $T(2,3)$

$$\frac{(1-q^{12})}{(1-q^4)(1-q^6)} \cdot \frac{(1-q^{26})}{(1-q^{13})}$$

$\approx \Delta$ for $T(2,13)$

<u>Corollaries:</u> (a) All roots of Δ for <u>algebraic knots</u> are roots of 1.

(b) All coefs of $\frac{\Delta(q)}{1-q}$ are 0 or 1

↗

$1-q$ are

Rmk One can compute HF homology of algebraic knots from S.

(c) For algebraic knots, the following contain equivalent info:

$$\begin{tikzcd}
\left(\begin{array}{c}\text{Puiseaux}\\ \text{pairs}\end{array}\right) \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \left(\begin{array}{c}\text{topological}\\ \text{type} = \text{iterated}\\ \text{cable}\end{array}\right) \arrow[d, Leftrightarrow] \\
S \text{ semigroup} \arrow[r, Leftrightarrow] & \Delta(q)
\end{tikzcd}$$

Rmk There are analogues of these results for algebraic links with many components

$S \rightsquigarrow$ multi-dimensional semigroup $\subset \mathbb{Z}^r$

$\Delta \rightsquigarrow \Delta(q_1 \ldots q_r)$

Thm (Yamamoto) In general, multi-variable Alexander polynomial $\Delta(q_1 \ldots q_r)$

Alexander polynomial $\Delta(q_1 \dots q_r)$
is a complete top. invariant of
algebraic links with r components.