Meeting 2/25

Thursday, February 25, 2021 2:53 PM

$Bl_0 \mathbb{C}^2$

<u>Very important lemma</u>

$$\boxed{E \circ E = -1}$$

<u>Proof</u> Consider the function x on $\mathbb{C}^2$. Its strict transform is C, total transform $E + C$.

(note: multiplicity = 1)

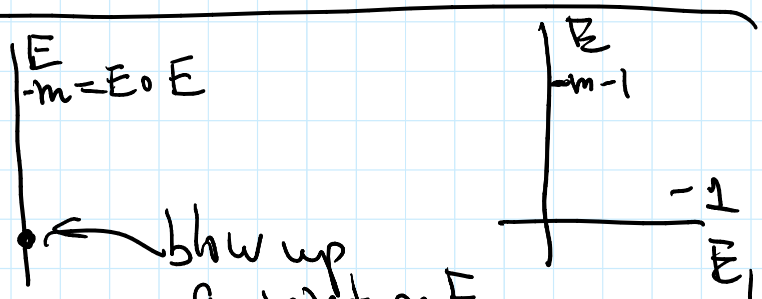
$$[\{x=0\}] = [E] + [C] = [\{x = \varepsilon\}]$$

Observe $\{x = \varepsilon\} \cap E = \emptyset$

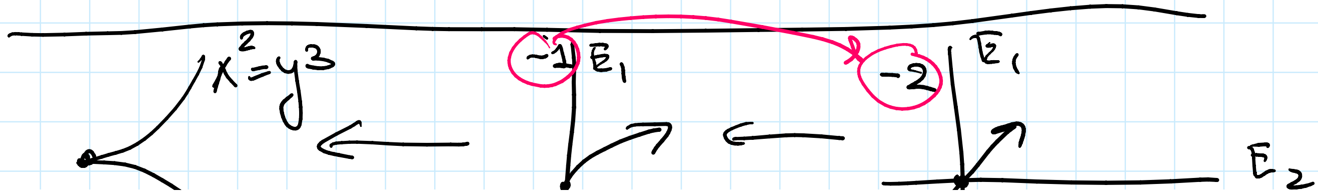
$$([E] + [C]) \circ [E] = 0$$

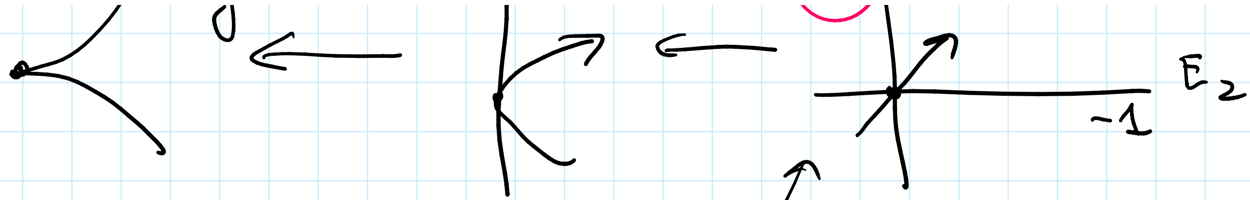
$$\underset{-1}{E \circ E} + \underset{1}{C \circ E} = 0$$

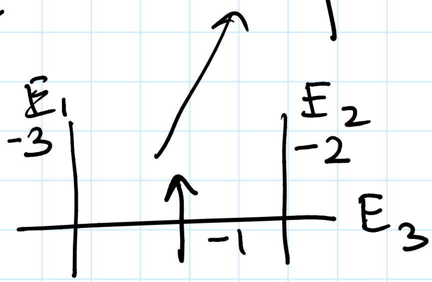
<u>Fact</u>



After blow up the self intersection of E decreases by 1







Intersection matrix $(E_i \circ E_j) = M$

$$\begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} = M$$

$$E_i \circ E_j = \begin{cases} -m, & i=j \\ 1 & \text{if } E_i \text{ intersects } E_j \\ 0 & \text{otherwise.} \end{cases}$$

$[E_i]$ give basis in H_2(resolution)

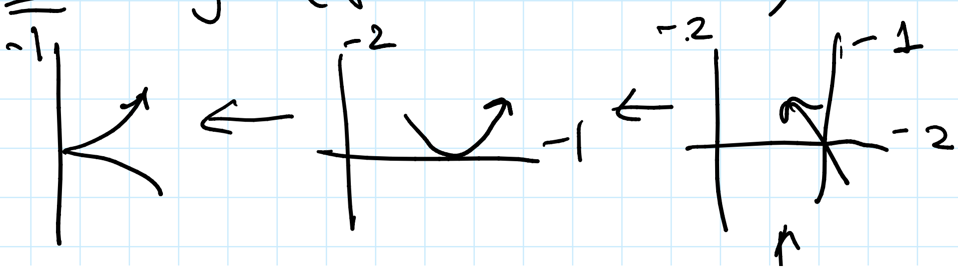
Thm $\det M = \pm 1$ and $-M^{-1}$ has all strictly positive entries!

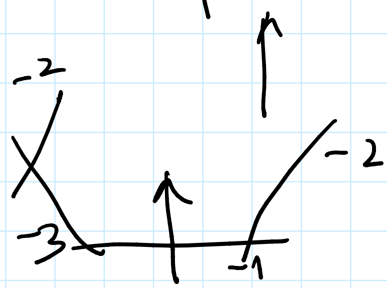
Ex

$$\det \begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} = -6+2+3 = -1$$

$$-\begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix}$$

Ex $x^2 = y^5$ (from last time)





$$M = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \qquad -M^{-1} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 4 & 10 & 5 \\ 1 & 2 & 5 & 3 \end{pmatrix}$$

Multiplicities from last time!

__Proof__ Suppose that f is a function such that:

- The strict transform of f intersects E_i at v_i points (transversally)
- The total transform of f has multiplicity n_i on E_i

$$[\{f=0\}] = \sum n_i [E_i] + [C]$$

$[C]$ ← strict transform.

On the one hand,

$$[\{f=0\}] \cdot [E_j] = [\{f=\varepsilon\}] \cdot [E_j] = 0$$

$$\| $$

$$\sum n_i E_i \cdot E_j + [C] \cdot E_j \ldots$$

$$\sum n_i E_i \cdot E_j + [C] \cdot E_j = 0$$

$$\sum n_i \cdot m_{ij} + v_i = 0$$

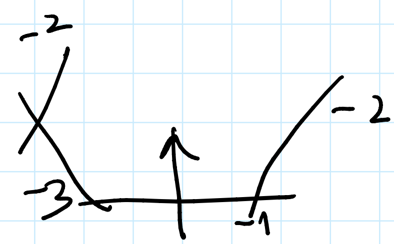
$$M \cdot \bar{n} + \bar{v} = 0$$

$\bar{n}$ = vector of n_i $\quad$ $\bar{v}$ = vector of v_i

$$\boxed{\bar{n} = -M^{-1} \cdot \bar{v}} > 0$$

All entries of $\bar{n}$ are > 0 for all choices of $\bar{v}$.

Ex $\quad \bar{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$



$$M = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \qquad -M^{-1} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 4 & 10 & 5 \\ 1 & 2 & 5 & 3 \end{pmatrix}$$

Multiplicities from

$$\bar{n} = -M^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \\ 5 \end{pmatrix} \leftarrow \text{multiplicity}$$

blowdown

$\rightsquigarrow$ some 4-component curve on $\mathbb{C}^2$

$\rightsquigarrow$ some f

$$\bar{n} = -M^{-1} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \ldots \quad \rightsquigarrow \text{ same } f$$

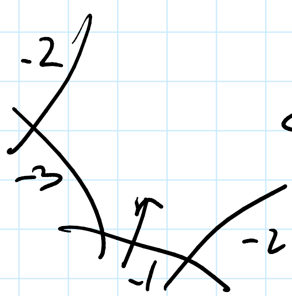
__Shape of resolution__:

Dual graph of resolution:

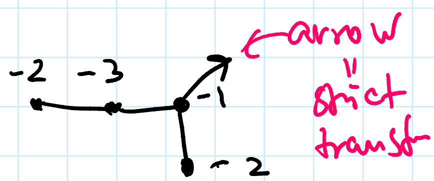
vertices $= E_i$ edges $\doteq E_i \circ E_j = 1$

(need to keep track of $E_i \circ E_i$)

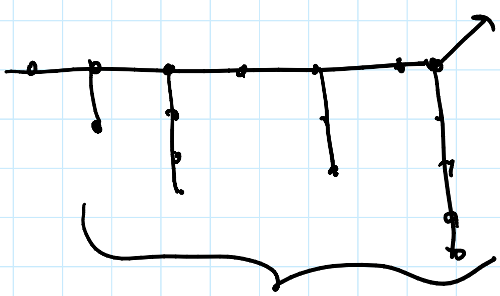
__Ex__



$\longleftarrow \cdots \longrightarrow$



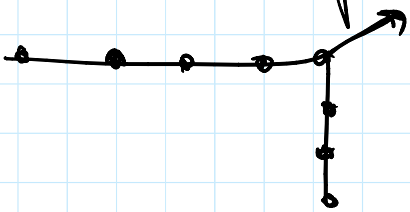
__Fact__ If $\{f=0\}$ is irreducible with d Puiseaux pairs, then the resolution tree has the shape



d legs

The length of legs, intersection matrix is controlled by Puiseaux pairs

Ex One Puiseaux pair



$x^a = y^b \qquad \gcd(a,b) = 1 \qquad a > b$

Blow up once: $y = xu$

$x^a = x^b u^b \qquad x^b(x^{a-b} - u^b) = 0$

E with multiplicity b

strict transform has Puiseaux pairs $(a-b, b)$

This is Euclidean algorithm!

$(a,b) \longrightarrow (a-b, b)$

Number of nodes in the resolution tree = number of steps in the Euclidean algorithm from (a,b) to $(1,0)$

$(a, a+1) \to (a,1) \to \underbrace{(a-1,1) \to (a-2,1) \to \cdots (0,1)}_{a \text{ steps}}$

Fact $S^3_d(K)$ = Dehn surgery of S^3 along K

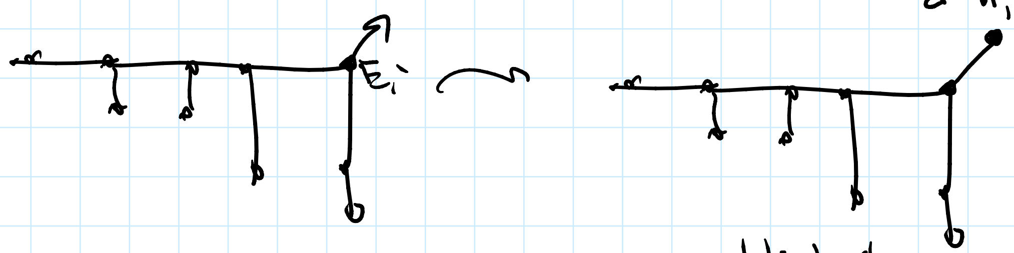
= some complicated 3-manifold

What is it?

– some complicated 3-manifold

What is it?

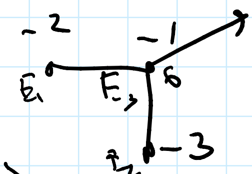
<u>Answer</u>: If K algebraic,



n_i = multiplicity on E_i

<u>Ex</u> $S^3_d(T_{2,3})$

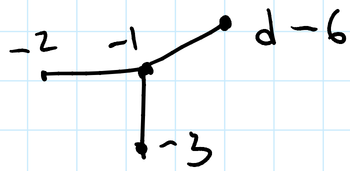
$T_{2,3} \longleftrightarrow \{x^2 = y^3\}$



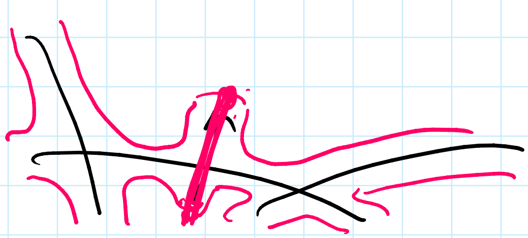
$$-\begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$n_3 = 6$

$S^3_d(T_{2,3})$



<u>Ex</u> $S^3_d(K)$ is reducible iff $d = n_i$



$\partial N(E_1 \cup E_2 \cup E_3 \cup C) = S^3$