Meeting 2/4

Thursday, February 4, 2021 3:02 PM

$\{f(x,y)=0\}$ plane curve singularity

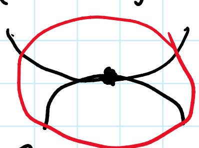
<u>Def</u> f is <u>reducible</u> if $f=g(x,y)\,h(x,y)$

Irreducible otherwise

<u>Fact</u>: Any f can be uniquely decomposed into irreducible factors $f=f_1\cdot f_2\cdot\ldots\cdot f_r$

<u>Ex</u>: $f=x^4-y^6=(x^2-y^3)(x^2+y^3)$

$\{f=0\}=\{f_1=0\}\cup\{f_2=0\}\cup\ldots\cup\{f_r=0\}$



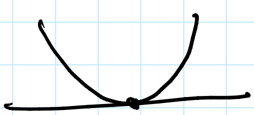
← two cusps, $x^2-y^3=0$

$x^2+y^3=0$

<u>Fact</u>: The link of f has r connected components = links of f_i

Linking number between components?

Later today.



$y(y-x^2)=0$

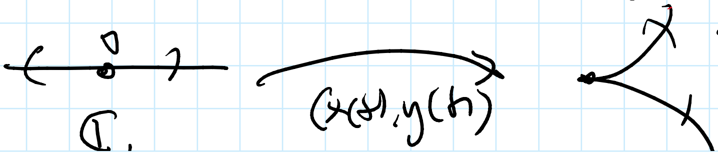
$\{y=0\}\cup\{y=x^2\}$

two unknots

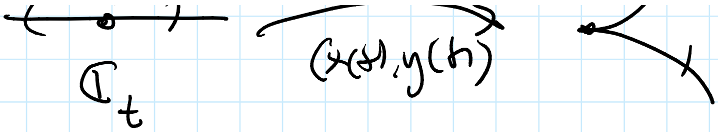
$lk=2$

<u>Thm</u> (Newton, Puiseaux ...) Any <u>irreducible</u> plane curve singularity $\{f(x,y)=0\}$ can be parametrized $x=x(t)$, $y=y(t)$ ← some power series in t

$f(x(t),y(t))=0$



$(x(t),y(t)):(\mathbb{C},0)$



$(x(t), y(t)) : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$

line $= S^1 \longrightarrow$ link of f image $= \{f = 0\}$

$\shortparallel$
knot $=$ image of S^1 in S^3

<u>Ex</u> $\{x^m = y^n\}$ $\gcd(m,n) = 1$

$x = t^n \quad y = t^m$

<u>Recall</u> $\gamma(t) =$ powers series $= a_k t^k +$ higher order

<u>Order of γ</u>: $\mathrm{Ord}\,\gamma(t) = k$ $a_k \neq 0$

<u>Plan</u>: simplify parametrization $(x(t), y(t))$ as much as possible + extract top. information

<u>Step 1</u> $\begin{cases} x(t) = a_p t^p + \dots & a_p \neq 0 \\ y(t) = b_q t^q + \dots & b_q \neq 0 \end{cases}$ Can assume $p \leq q$

$$x(t) = a_p t^p (1 + \dots) =$$
$$= (\hat{t})^p \quad \text{where } \hat{t} = t \cdot (1 + \dots)^{1/p} \sqrt[p]{a_p}.$$

powers series in t.

Change variable $t \longrightarrow \hat{t}$ and get

since $\left.\dfrac{\partial \hat{t}}{\partial t}\right|_{t=0} \neq 0$

$\hat{t} \leftrightarrow t$ is invertible change of variable

$$\begin{cases} x(t) = t^p \\ y(t) = b_q t^q + \dots \end{cases}$$

<u>Step 2</u>: Delete all terms in $y(t)$ with exponents divisible by p:

$$(x, y) \longleftrightarrow (x, y - b_p x - b_{2p} x^2 - \dots)$$

$$\begin{cases} x(t) = t^p \\ \dots \end{cases}$$

no exp.

$$\begin{cases} x(t) = t^p \\ y(t) = b_q t^q + \dots \end{cases}$$ ← no exp. divisible by p

$q > p$, not divisible by p

Def: (p,q) = first Puiseaux pair

If $GCD(p,q) = 1$, we are done.

In this case $\boxed{\text{link of } f = (p,q) \text{ torus knot}}$

Else, Step 3 If $GCD(p,q) = d > 1$

find the first $q_1 > q$ such that $b_{q_1} \neq 0$
and q_1 not divisible by d

Ex
$$\begin{cases} x = t^4 \\ y = t^6 + t^7 \end{cases}$$

$p = 4$, $q = 6$, $d = GCD(4,6) = 2$

$q_1 = 7$

(d, q_1) = second Puiseaux pair.

$d_1 = GCD(d, q_1)$

If $d_1 = 1$, stop. In this case,

$\boxed{\text{link of } f = \text{cable of } \left(\frac{p}{d}, \frac{q}{d}\right) \text{ torus knot}}$

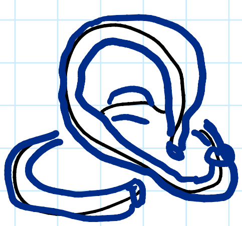
$(4,6) \to (2,3)$ torus knot

$(2,7) \longrightarrow$ Link = $(2,13)$ cable of $\underbrace{(2,3)}_{\frac{p}{d}\ \frac{q}{d}}$ torus knot,

d

$q_1 + \dots$

first Puiseaux pair



K = knot

T = tubular neighborhood of K

∂T = torus

T = tubular neighborhood of K

∂T = torus

Can draw a torus knot ${}^{(a,b)}$ on that torus $\longrightarrow$ (a,b) cable of K.

In this case $K = (2,3)$, draw $(2,13)$ torus knot on ∂T.

Else, Step 4: continue..

Key fact: (a) Puiseaux pairs completely control the topology of f (if f is irreducible)

(b) Link of f = iterated cable of $\left(\frac{p}{d}, \frac{q}{d}\right)$ torus knot

Ex 3 Puiseaux pairs

$$K_1 \rightsquigarrow K_2 \rightsquigarrow K_3$$

$K_1 = T\left(\frac{p}{d}, \frac{q}{d}\right)$ = torus knot

K_2: cable of K_1 determined by 2nd Puiseaux pair

K_3: Cable of K_2 determined by 3rd Puiseaux pair

(t^m, t^n)

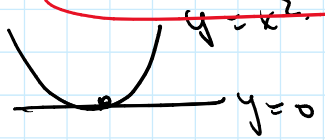
$t = e^{2\pi i \varphi}$ $\quad \left(e^{2\pi i m\varphi}, e^{2\pi i n\varphi}\right) \in |z_1|^2 + |z_2|^2 = 1 \leftarrow S^3$

$\rightarrow |z_1| = 1 \quad |z_2| = 1$ = torus in

Linking #: $K = \{f = 0\}$ parametrize $(x(t), y(t))$ $\qquad L = \{g = 0\}$

parametrize
$(x(t), y(t))$
as above

Claim $\boxed{lk(K,L) = \operatorname{Ord} g(x(t),y(t))}$



$\longleftarrow (t,0)$

Fact from algebra: symmetric in f and g.

$g = x^2 - y \Rightarrow g(x(t),y(t)) = t^2$ $\quad \operatorname{Ord} = 2$

Ex $\{x^4 = y^6\}$ $\quad (x^2-y^3)(x^2+y^3) = 0$

parametrize $(x,y) = (t^3, t^2)$

plug into second eqn.

$g(x,y) = x^2 + y^3$

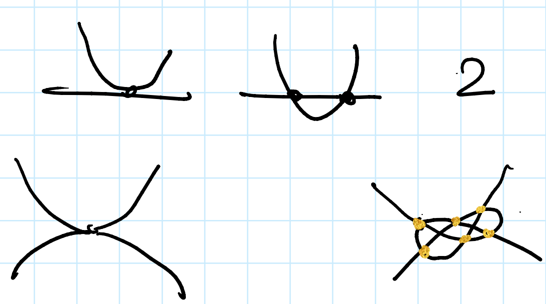
$g(t^3,t^2) = t^6 + t^6 = 2t^6$

Linking # $= \operatorname{Ord} = 6$

Conclusion: these are two trefoils with linking # 6.

Note $lk(K,L) =$ # intersection points of small deformations of $f=0$, $g=0$

Ex



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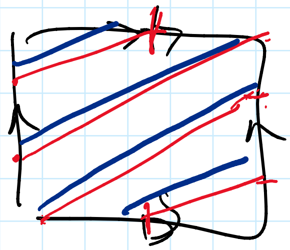
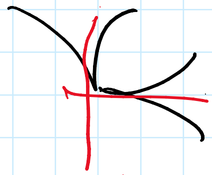
6

self-intersections do not count.

Torus



(2, 3) torus knot



$(x^2-y^3)(x^3-y^2)$

$lk = 4$