Meeting 1/14

Thursday, January 14, 2021 12:54 PM

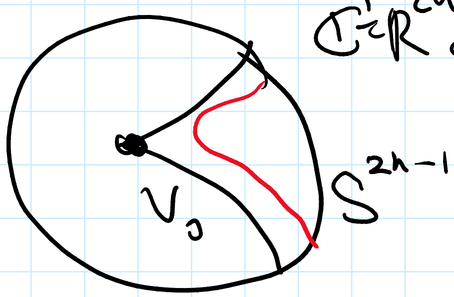
$f : \mathbb{C}^n \to \mathbb{C}$ polynomial in n complex variables

critical pt $= \{ \frac{\partial f}{\partial z_1} = \dots = \frac{\partial f}{\partial z_n} = 0 \}$

isolated singularity $\{ f = 0 \}$

$V_\varepsilon = \{ f = \varepsilon \} \cap B^{2n}_\delta$ ← Milnor fiber

$\mathbb{C}^n \cong \mathbb{R}^{2n}$ small ball



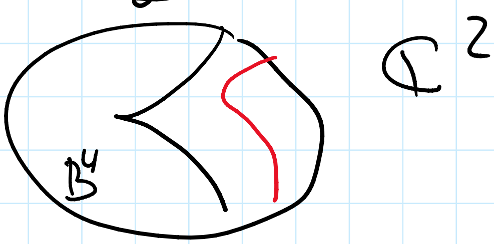
If $0 < \varepsilon \ll \delta \ll 1$

V_ε is smooth complex mfd

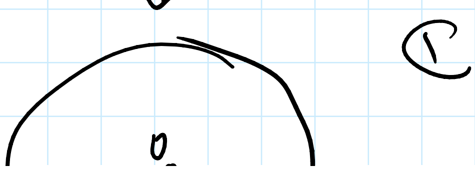
$\dim V_\varepsilon = 2n-2$

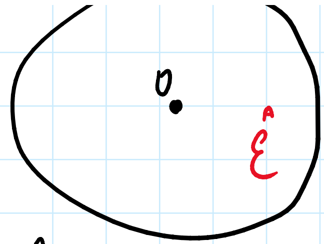
<u>Goal</u>: Understand the topology of V_ε

$\{ \underbrace{x^2 - y^3} = 0 \}$ crit. pt $(0,0)$



$\downarrow f$





(1) Deform $f \rightsquigarrow \tilde{f}$

<u>Fact</u>: $\{f=\varepsilon\} \cap B_\delta^{2n} \approx \{\tilde{f}=\varepsilon\} \cap B_\delta^{2n}$

assuming that $\tilde{f}$ is "close" to f

<u>Ex</u> $f = x^2 - y^3$

$$\tilde{f} = x^2 - y^3 + \lambda y \qquad \underline{\lambda \text{ small}}$$

Critical points of $\tilde{f}$:

$$\frac{\partial \tilde{f}}{\partial x} = 2x \qquad \frac{\partial \tilde{f}}{\partial y} = -3y^2 + \lambda$$

$$x = 0, \quad y = \pm\sqrt{\frac{\lambda}{3}}$$

- 2 critical points for $\tilde{f}$, close to $(0,0)$ = critical point for f
- Nondegenerate: $\det \underbrace{\left(\frac{\partial f}{\partial z_i \partial z_j}\right)}_{H} \neq 0$

$$H = \begin{pmatrix} 2 & 0 \\ 0 & -6y \end{pmatrix} \qquad \det H \neq 0 \text{ at } \left(0, \pm\sqrt{\tfrac{\lambda}{3}}\right)$$

at $(0, \pm\sqrt{3})$

- Critical values are distinct
$$\tilde{f}\left(0, \sqrt{\tfrac{\lambda}{3}}\right) \neq \tilde{f}\left(0, -\sqrt{\tfrac{\lambda}{3}}\right)$$

<u>Def</u> $\tilde{f}$ is called a <u>morsification</u> of f, if
- $\tilde{f}$ is small deformation of f $\qquad \tilde{f} = f + \lambda g$, λ small
- $\tilde{f}$ have nondegenerate critical points
- Critical values for $\tilde{f}$ are distinct.

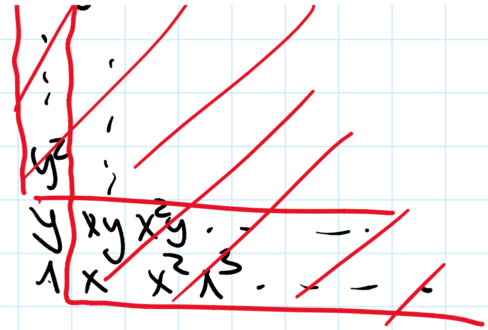
<u>Thm</u> (Milnor) The number of critical points for such $\tilde{f}$ does not depend on particular choice of $\tilde{f}$ and equals
$$\mu = \dim \frac{\mathbb{C}[[z_1, \dots z_n]]}{\left(\frac{\partial f}{\partial z_1}, \dots \frac{\partial f}{\partial z_n}\right)} = \text{Milnor number}$$

ideal generated by $\frac{\partial f}{\partial z_i}$

<u>Ex</u> $f = x^2 - y^3$
$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = -3y^2$$

$$\frac{\mathbb{C}[[x,y]]}{\dots} = \operatorname{Span}\langle 1, y\rangle$$

$$\frac{\mathbb{C}[[x,y]]}{(2x,-3y^2)} = \text{Span}\langle 1, y\rangle$$



$$\mu = \dim \frac{\mathbb{C}[[x,y]]}{(2x,-3y^2)} = 2$$

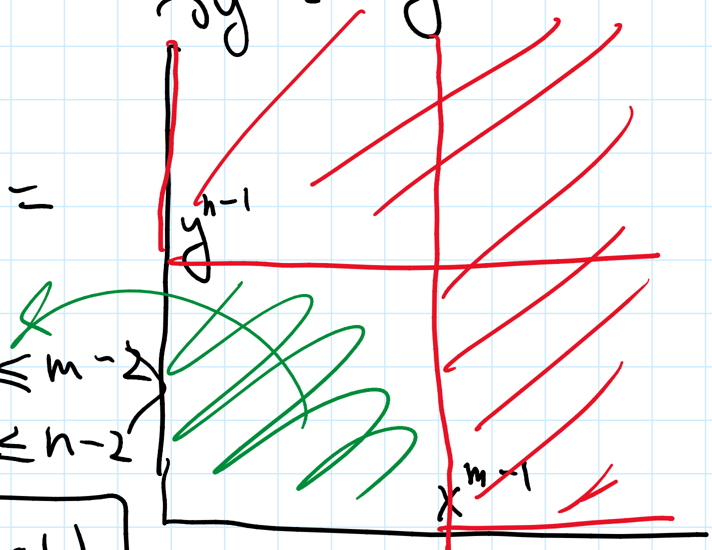
$\longleftrightarrow$ we have 2 nondegenerate critical points for $\tilde{f}$.

Ex $f = x^m - y^n$

$$\frac{\partial f}{\partial x} = m x^{m-1}$$

$$\frac{\partial f}{\partial y} = -n y^{n-1}$$

$$\frac{\mathbb{C}[[x,y]]}{(x^{m-1},y^{n-1})} =$$



$$\text{Span}\langle x^a y^b \mid a \le m-2,\ b \le n-2 \rangle$$

$$\boxed{\mu = (m-1)(n-1)}$$

$$\underbrace{x^m + \alpha_2 x^{m-2} + \dots + \alpha_m}_{} + \underbrace{y^n + \beta_2 y^{n-2} + \dots + \beta_n}_{} = \tilde{f}$$

Crit. points = { crit. points of φ_x } crit. point of ψ

Crit. points = $\left\{\begin{matrix}\text{crit. points of}\\ x^m+\dots+\alpha_m\end{matrix}\right\} \times \left\{\begin{matrix}\text{crit. point of}\\ y^n+\dots+\beta_n\end{matrix}\right\}$

$m-1$, α generic $\quad$ $n-1$ $\quad$ β gen.

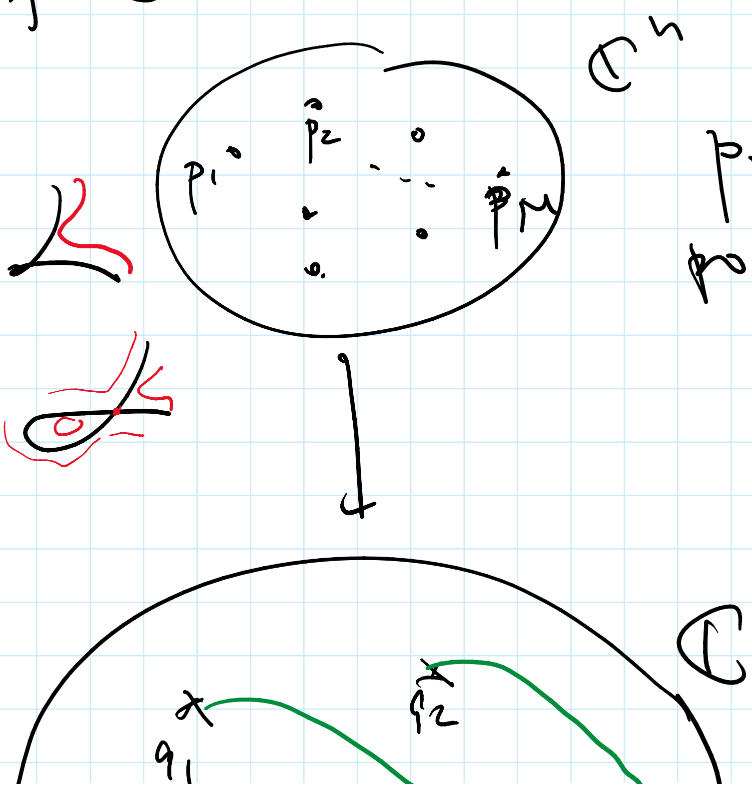
② Recall from last time:

$\tilde{f}$ has nondegenerate critical pt

$\Longleftrightarrow$ (complex Morse lemma) can change variables to $\tilde{f} = z_1^2 + \dots + z_n^2$

$V_\varepsilon \cong TS^{n-1}$ ← vanishing cycle shrinks as $\varepsilon \to 0$

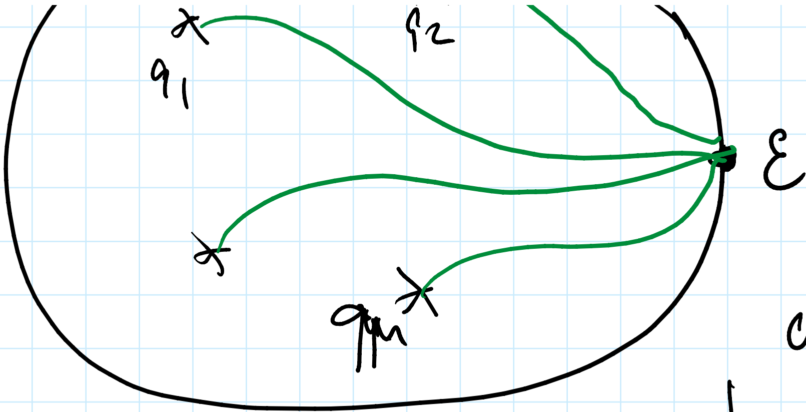
$\tilde{f}: \mathbb{C}^n \to \mathbb{C}$



$p_1 \dots p_\mu$ = critical points of $\tilde{f}$

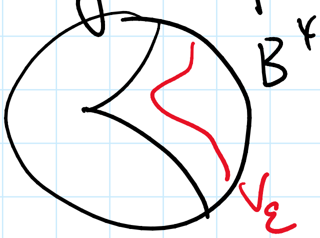
$q_1 \dots q_\mu$ = critical values

Pick ε at

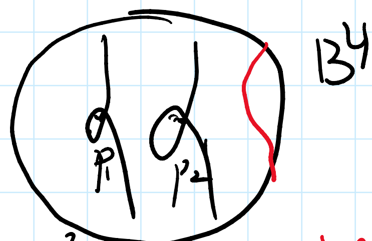


Pick ε at the boundary of a circle, and connect it with $q_1 \ldots q_m$ by a system of paths (non-intersecting)

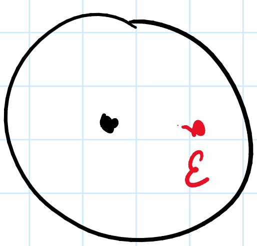
$x^2 - y^3 = f$

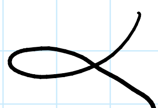


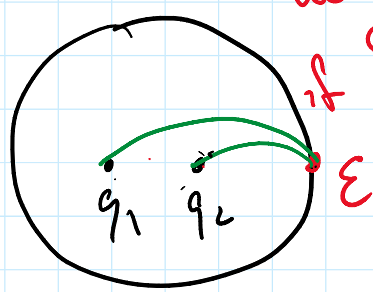
$x^2 - y^3 + \lambda y = \hat{f}$



$x^2 - y^3 + \lambda y = q_i$







V_ε did not change if $\lambda << \varepsilon$

q_1, q_2 near 0

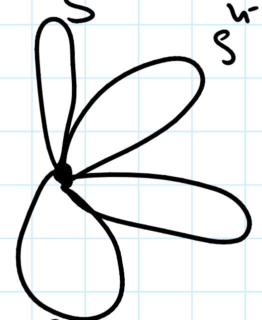
Main construction:

Near q_i we have a S^{n-1} vanishing cycle which vanishes as we approach q_i

We ...

We transport this cycle along the path to ε

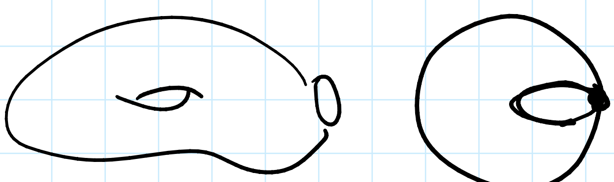
Main Thm (Milnor) The resulting cycles in V_ε span $H_{n-1}(V_\varepsilon)$.



Moreover, V_ε is homotopy equivalent to a bouquet of μ $(n-1)$ dimensional spheres.

$$H_*(V_\varepsilon) = \begin{cases} H_0 = \mathbb{Z} \\ H_{n-1} = \mathbb{Z}^\mu \end{cases}$$

Ex $x^2 - y^3 = \varepsilon$ torus with boundary



$S^1 \vee S^1$

Ex $n=2$ V_ε = surface with boundary

$\rightsquigarrow$ homotopy equivalent to a bouquet of circles

$$\#\text{circles} = 2g + r - 1 = \mu$$

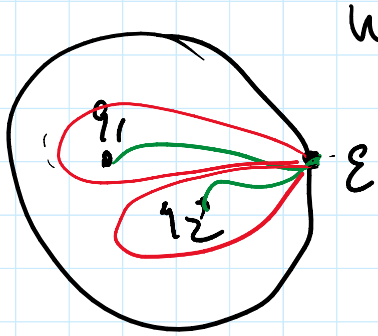
r = # boundary components

If we know r and

If we know r and # boundary components μ, can compute the genus.

More complicated questions (later in the quarter)

- Intersection form on $H_{n-1}(V_\varepsilon)$
- Monodromy (as ε goes around 0)
- (maybe) Monodromy group



move ε around different q_i

All these capture more than just the Milnor number.