Singular curves: blow-ups

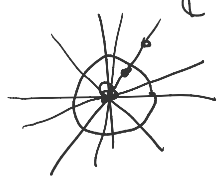
Thursday, January 21, 2021 3:13 PM

Projective space $\mathbb{CP}^N = \mathbb{C}^{N+1} - \{0\} / \sim$

$(z_0, z_1, \dots, z_N) \sim (\lambda z_0, \lambda z_1, \dots, \lambda z_n)$

"homogeneous coords" $= \{ [z_0 : z_1 : \dots : z_N] \mid (z_0, z_1, \dots, z_N) \in \mathbb{C}^{N+1} \setminus 0 \}$ ← up to $\sim$

$\mathbb{C}^2$



① Affine charts: Pick any coord (say z_0)

$\mathbb{CP}^N = \{ [z_0 : z_1 : \dots : z_N] \mid z_0 \neq 0 \} \cup \{ [z_0 : z_1 : \dots : z_N] \mid z_0 = 0 \}$

$\tilde{z}_i = \frac{z_i}{z_0}$ $\quad = \underbrace{\{ [1 : \tilde{z}_1 : \dots : \tilde{z}_N] \mid (\tilde{z}_1, \dots, \tilde{z}_N) \in \mathbb{C}^N \}}_{\cong \mathbb{C}^N \text{ Affine chart}} \cup \underbrace{\{ [0 : z_1 : \dots : z_N] \}}_{\cong \mathbb{CP}^{N-1}}$

$\mathbb{CP}^N = \mathbb{C}^N \cup (\mathbb{C}^{N-1} \cup (\mathbb{CP}^{N-2}))$

Affine charts cover $\mathbb{CP}^N$

$\mathbb{CP}^1 = \mathbb{C} \cup \mathbb{CP}^0_{\approx pt}$ 1pt compactification of $\mathbb{R}^2$

$\cong S^2$

② Alg curves in $\mathbb{CP}^2$

zero set of a homogeneous polynomial is welldefined in homogen coords

e.g. $[z_0 : z_1 : z_2]$ $\quad \{ z_0^2 + z_1 z_2 + 3 z_0 z_1 = 0 \}$

$\shortparallel$

$[\lambda z_0 : \lambda z_1 : \lambda z_2]$ $\quad (\lambda z_0)^2 + (\lambda z_1)(\lambda z_2) + 3(\lambda z_0)(\lambda z_1) \stackrel{?}{=} 0$

$\shortparallel$

$\lambda^2 (z_0^2 + z_1 z_2 + 3 z_0 z_1) = 0$

In affine chart $(z_0 = 1)$ $\quad \{ 1 + z_1 z_2 + 3 z_1 = 0 \}$

$\underbrace{\hspace{3cm}}$

non homog poly in $(z_1, z_2) \in \mathbb{C}^2$

Blow-up at a smooth point

Localized operation so assume blow up $(0,0)$ in $\mathbb{C}^2$
$\mathbb{C}^2 = \{(x,y)\}$ (or $0 \in \mathbb{C}^n$)

$$\mathrm{Bl}_0(\mathbb{C}^2) = \{((x,y),[u:v]) \in \underbrace{\mathbb{C}^2 \times \mathbb{CP}^1} \mid \underbrace{xv - yu = 0}\}$$

$$\pi \downarrow \qquad \qquad \downarrow$$

$$\mathbb{C}^2 \qquad \qquad (x,y)$$

$xv - yu = 0 \quad xv = yu$

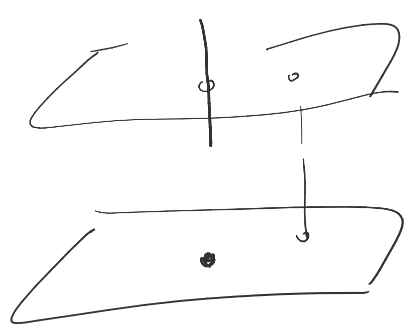
$\underline{v = \frac{y}{x} u}$ or $u = \frac{x}{y} v$

For $(x,y) \neq (0,0)$ $\quad \pi^{-1}((x,y)) = \{((x,y),[u : \frac{y}{x} u])\} \leftarrow$ one point

$\{((x,y),[x:y])\}$

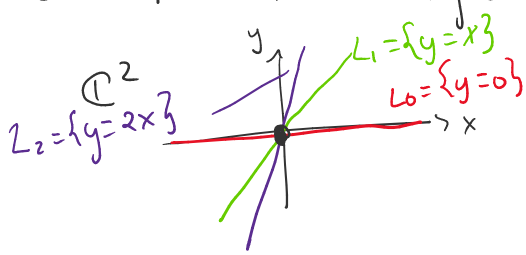
exceptional curve $\rightarrow$ $\pi^{-1}((0,0)) = \{((0,0),[u:v]) \mid [u:v] \in \mathbb{CP}^1\}$
exceptional divisor

$0 \cdot v - 0 \cdot u = 0$
always true



What does blow up do to submflds (complex curves)?

Example:



In $\mathrm{Bl}_0(\mathbb{C}^2)$ $\quad \underline{\pi^{-1}(L_0)} = \{((x,y),[u:v]) \mid \begin{matrix} xv - yu = 0 \\ y = 0 \end{matrix}\}$

$\pi \downarrow$ $\quad$ Total transfor

In $\begin{array}{c} Bl_0(\mathbb{C}^2) \\ \pi \downarrow \\ \mathbb{C}^2 \end{array}$ $\quad \underbrace{\pi^{-1}(L_0)}_{\text{Total transform of } L_0 \;\nearrow\; \overline{L_0}} = \{((x,y),[u:v]) \mid y=0\}$

$= \{((x,0),[u:v]) \mid xv=0\}$

$= \{((x,0),[1:0])\} \cup \{((0,0),[u:v])\}$

$\tilde{L}_0 \longrightarrow$ proper transform of L_0 $\qquad$ exceptional curve $\leftarrow E$

Note $\tilde{L}_0 \cap E$ is a single point $\{((0,0),[1:0])\}$

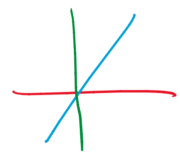
Similarly $\quad \pi^{-1}(L_1) = \{((x,x),[u:v] \mid xv - xu = 0\}$ $\quad x=0$ or $v-u=0$

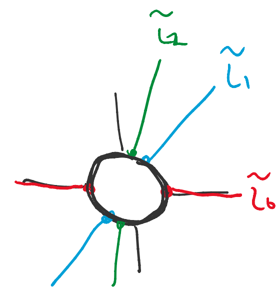
$= \{(\underbrace{(x,x),[1:1]}_{\tilde{L}_1})\} \cup \{((0,0),[u:v])\}$

exc curve

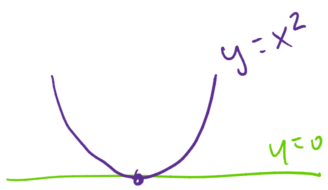
again $\tilde{L}_1 \cap E = \{((0,0),[1:1])\}$

similarly w/ L_2

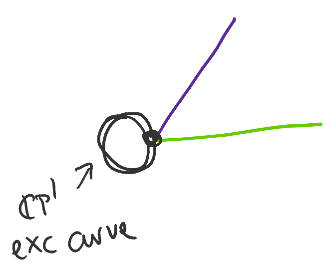




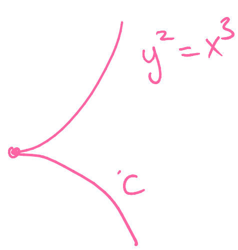
Moral: Lines approaching 0 from diff directions are separated after taking proper xforms in $Bl_0(\mathbb{C}^2)$



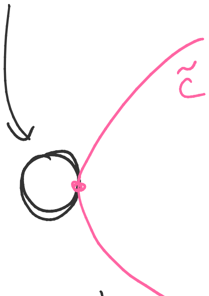
blowup $\longrightarrow$



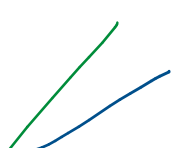
Blowup makes curves less tangent



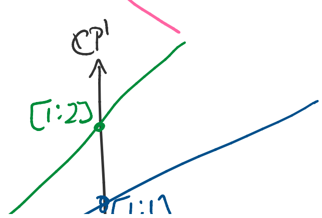
blowup $\longrightarrow$

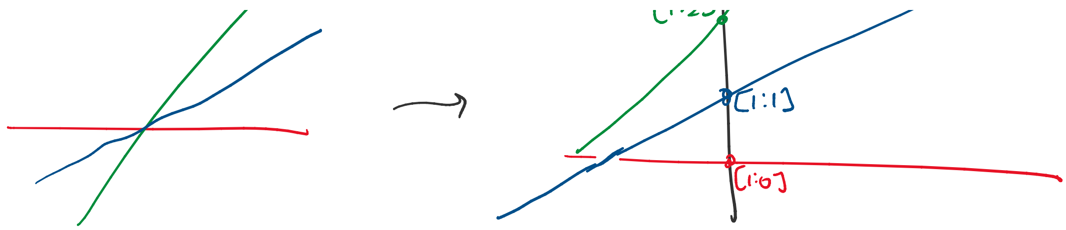


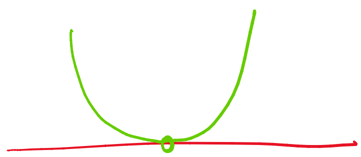
Blow-up makes curves less singular

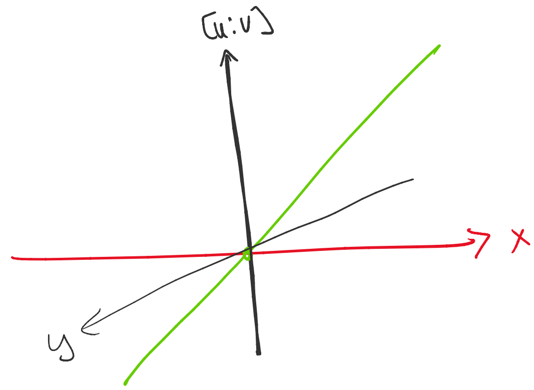


$\longrightarrow$




[1:1]
[1:0]




[u:v]
x
y