Singular curves: blow-ups continued

Thursday, January 28, 2021 3:11 PM

$$\begin{array}{ccc} \tilde{X} & Bl_0(\mathbb{C}^2) & \left(\begin{array}{c} Bl_0(\mathbb{C}^n) \\ \pi \downarrow \\ \mathbb{C}^n \end{array} \quad \pi^{-1}(0) \cong \mathbb{CP}^{n-1} \right) \\ \pi \downarrow & \pi \downarrow & \\ X & \mathbb{C}^2 & \end{array}$$

$\pi : Bl_0(\mathbb{C}^2) \setminus \pi^{-1}(0) \longrightarrow \mathbb{C}^2 \setminus 0$ is 1-1 isomorphism

$\pi^{-1}(0) \cong \mathbb{CP}^1$

$Bl_0(\mathbb{C}^2) = \{ ((x,y), [u:v]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid xv - yu = 0 \}$

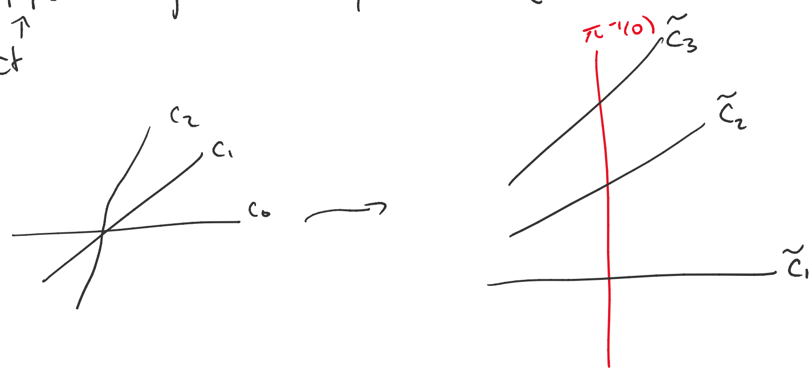
If have some complex curves C_1, C_2 in $\mathbb{C}^2$

Total transform : $\bar{C}_i = \pi^{-1}(C_i) \subset Bl_0(\mathbb{C}^2)$

Proper transform : $\tilde{C}_i = \overline{\pi^{-1}(C_i - 0)}$

strict

Example 1 (last time)

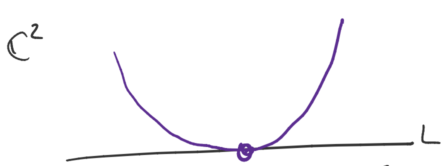


Example 2:

$\{ y(y-x^2) = 0 \} = \{ y = 0 \} \cup \{ y - x^2 = 0 \}$

$\qquad\qquad\qquad\qquad\quad \| \qquad\qquad\quad \|$

$\qquad\qquad\qquad\qquad\quad L \qquad\qquad\quad Q$



blow up $\longrightarrow$

$\bar{L} = \left\{ ((x,y), [u:v]) \;\middle|\; \begin{array}{c} xv - yu = 0 \\ y = 0 \end{array} \right\}$

$\bar{Q} = \left\{ ((x,y), [u:v]) \;\middle|\; \begin{array}{c} xv - yu = 0 \\ y - x^2 = 0 \end{array} \right\}$

Choose 1 coord chart at a time :

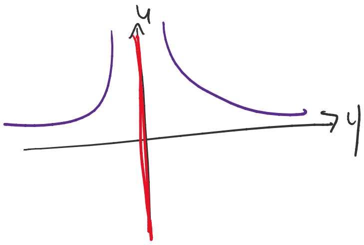
$\underline{v = 1 \text{ chart}}$ Blowup eqn : $\underline{x - yu = 0} \longrightarrow \underline{x} = yu$

Coordinates on $Bl_0(\mathbb{C}^2)$ in this chart are (y,u) (remember $x = yu$, $v = 1$)

Coordinates on $Bl_0(\mathbb{C}^2)$ in this chart are (y,u) (remember $x = yu$, $v=1$)

$\overline{L} = \{(y,u) \mid y = 0\}$ ← exceptional divisor

$\overline{Q} = \{(y,u) \mid y - (yu)^2 = 0\} = \{(y,u) \mid y(1-yu^2) = 0\}$



$= \{(y,u) \mid y=0\} \cup \underline{\{(y,u) \mid 1 - yu^2 = 0\}}$

exc divisor

no intersection in this chart

how proper transform $\tilde{Q}$ intersects this chart

In chart:

$u=1$ $\quad xv - y = 0$ solve for $y = xv$ use coords (x,v)

$\overline{L} = \{(x,v) \mid xv = 0\} = \{(x,v) \mid x=0\} \cup \underline{\{(x,v) \mid v=0\}}$

exc divisor

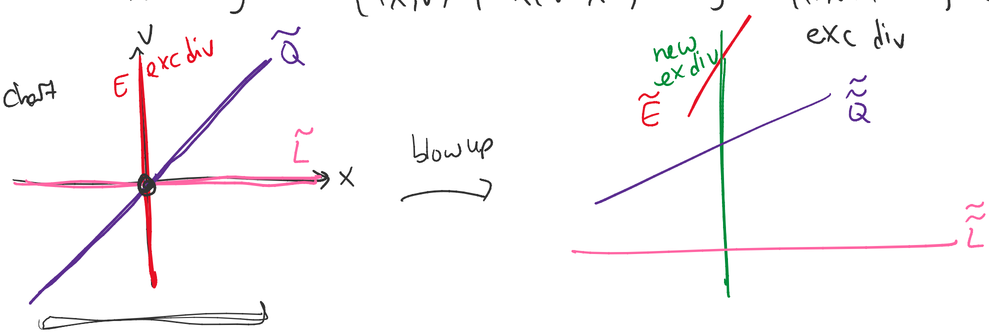
proper transform

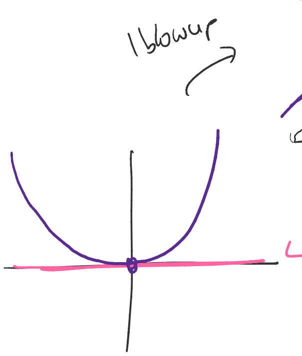
(intersect at $(0,0) = (x,v)$)

$\overline{Q} = \{(x,v) \mid xv - x^2 = 0\} = \{(x,v) \mid x(v-x) = 0\} = \{(x,v) \mid x=0\} \cup \{(x,v) \mid v - x = 0\}$

exc div $\qquad \tilde{Q}$

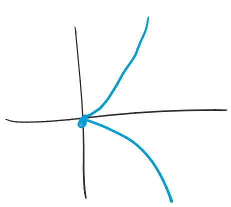
In (x,v) chart





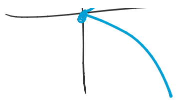
$\{y=0\} \cup \{y = x^n\}$

Example 2: $\quad C = \{x^3 - y^2 = 0\}$



$\xrightarrow{\text{blowup}}$ $\quad \overline{C} = \left\{ ((x,y),[u:v]) \;\middle|\; \begin{matrix} xv - yu = 0 \\ x^3 - y^2 = 0 \end{matrix} \right\}$

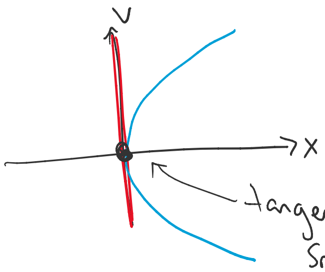
$u=1$ $\quad y = xv$ $\quad$ use coords (x,v) on this chart



$\underline{\underline{u=1}}$ $\quad y = xv \quad$ use coords (x,v) on this chart

$$\overline{C} = \{ (x,v) \mid x^3 - (xv)^2 = 0 \}$$
$$= \{ (x,v) \mid x^2(x - v^2) = 0 \}$$
$$= \underbrace{\{ (x,v) \mid x^2 = 0 \}}_{\text{exc } E} \cup \underbrace{\{ (x,v) \mid x - v^2 = 0 \}}_{\text{proper xform } \tilde{C}}$$

no singular points



tangency btwn smooth prop transform & exc curve

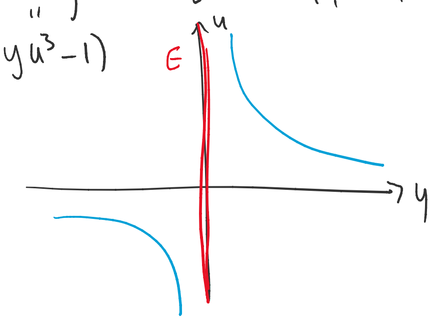
Idea: If exc curve is $\pitchfork$ to proper xform, then original curve in $\mathbb{C}^2$ is smooth whereas when exc curve is tangent, the curve is singular

$$\{ x^{k+1} - y^k = 0 \}$$

$\underline{v=1}$ chart: $\quad x - yu = 0 \quad x = yu \quad$ coords (y,u)

$$\overline{C} = \{ (y,u) \mid \underbrace{(yu)^3 - y^2}_{\substack{\shortparallel \\ y^2(yu^3-1)}} = 0 \} = \{ (y,u) \mid y^2 = 0 \} \cup \{ (y,u) \mid \underbrace{yu^3 - 1 = 0}_{y = 1/u^3} \}$$

In this chart:



Other things to try:

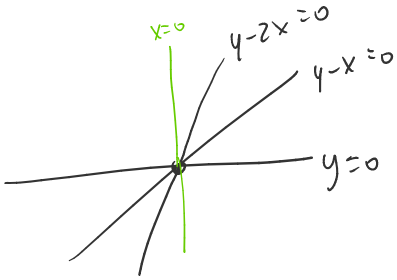
$y(y - x^k) = 0 \leftarrow$ become disjoint after k blow ups

$y^k - x^{k+1} = 0 \quad \leftarrow$ become smooth after 1 blow-up

$y^2 - x^5 = 0 \quad \leftarrow$ after one blow up still singular

looks like $\underbrace{y^2 - x^3 = 0}$

blow up again to get smooth



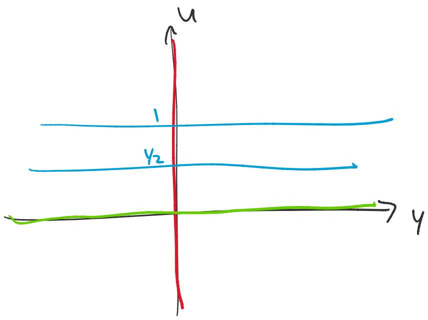
$V=1$: $x = yu$ $\quad (y,u)$

$\{y=0\}$ $\quad \{y - yu = 0\}$ $\quad \{y - 2yu = 0\}$

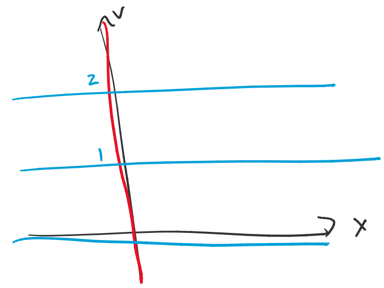
$u=1$: $y = xv$ $\quad (x,v)$

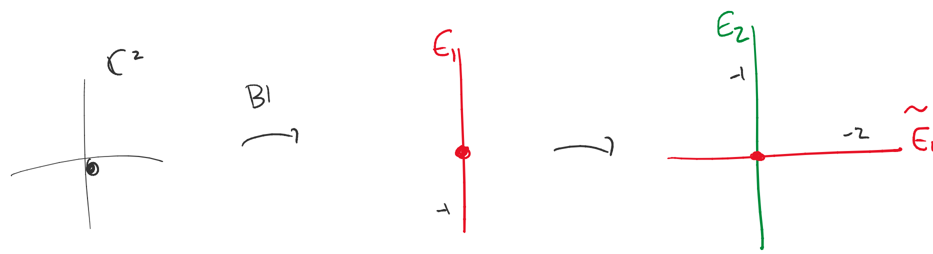
$\{xv = 0\}$ $\quad \{xv - x = 0\}$ $\quad \{xv - 2x = 0\}$

$\underline{V=1}$



$\underline{u=1}$





$(\tilde{E}_1 + E_2, E_2)$

$(\tilde{E}_1, E_2)$ $\quad \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{change basis}} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$