Meeting 1/7

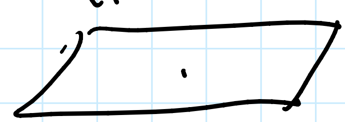
Thursday, January 7, 2021 12:42 PM

$\mathbb{C}^n \supset \{f(z_1 \ldots z_n) = 0\}$ hypersurface
polynomial

singular point $\{\frac{\partial f}{\partial z_1} = \ldots = \frac{\partial f}{\partial z_n} = 0\}$

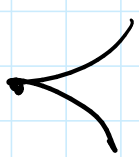
nonsingular : $\frac{\partial f}{\partial z_i} \neq 0$ for some i

Implicit function thm:
if $\frac{\partial f}{\partial z_i} \neq 0$ then we can change coordinates so that $\{f=0\} \sim \{z_i = 0\}$

$\Rightarrow$ no interesting topology. 

A singular point is <u>isolated</u> if there are no other singular points in some neighborhood.

<u>Ex</u> $\{z_1^2 = z_2^3\} \subset \mathbb{C}^2$

 $f = z_1^2 - z_2^3$ $\quad \frac{\partial f}{\partial z_1} = 2z_1$, $\quad \frac{\partial f}{\partial z_2} = 3z_2^2$

singular point $\{z_1 = z_2 = 0\}$ isolated.

$\{z_1^2 = z_2^3\} \subset \mathbb{C}^3$

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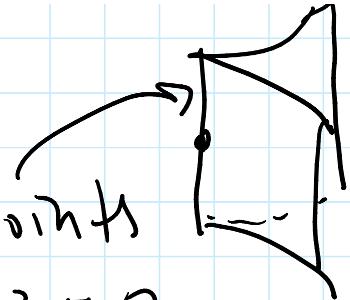
not isolated

line of singular points



$\{z_1 = z_2 = 0$, z_3 arbitrary$\}$

Brieskorn-Pham sing. $\{z_1^{a_1} + \dots + z_n^{a_n} = 0\}$ $\quad a_i \geq 2$

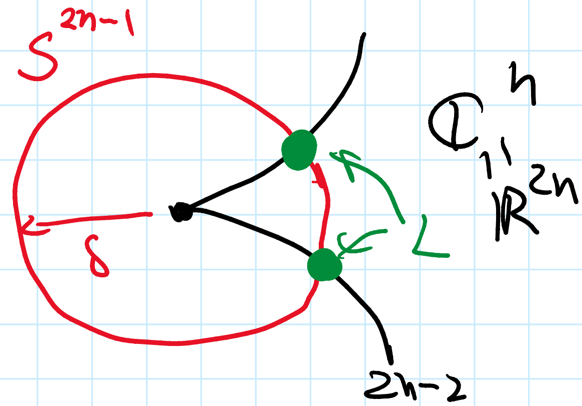
The only singular point $(0, \dots, 0)$ isolated.

<u>Thm</u> (Milnor) Suppose $\{f=0\}$ has isolated singularity at $(0, \dots, 0)$

Then for sufficiently small $\delta > 0$ the following holds:

sphere of radius δ

(a) The sphere S^{2n-1}_δ is transversal to $\{f=0\}$



(b) $L = S^{2n-1}_\delta \cap \{f=0\}$

<u>link of singularity</u>

smooth, $\dim = 2n-3$

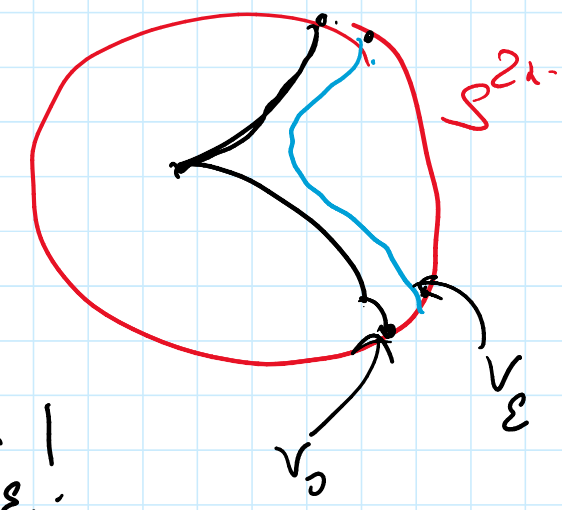
(c) $V_0 = \{f=0\} \cap B^{2n}_\delta$ is diffeomorphic the cone over L $(\Rightarrow$ contractible$)$

the cone over L ($\Rightarrow$ contractible)

(d) $V_\varepsilon = \{f = \varepsilon\} \cap B^{2n}_\delta$ Milnor fiber

for $0 < \varepsilon << \delta$ this is smooth $(2n-2)$-dim

manifold with $\partial V_\varepsilon \simeq L$



Ex $V_0\{z_1^2 - z_2^3\}$ singular

$V_\varepsilon = \{z_1^2 - z_2^3 = \varepsilon\}$ smooth

$$\frac{\partial f}{\partial z_1} = 2z_1, \quad \frac{\partial f}{\partial z_2} = 3z_2^2$$

Sing. point at $(0,0)$ not on V_ε!

Examples: $n = 2$ $\dim V_\varepsilon = 2$ $\dim L = 1$

link/knot (algebraic link)

V_ε = algebraic curve
= real smooth surface
in $\mathbb{D}^4$ boundary the link.

(we'll focus on this case).

Q: How to read off genus of V_ε?
Topological type of the link? Alexander polynomial?
from equation $f(z_1, z_2)$?

from equation $f(z_1, z_2)$!

<u>Ex</u> $n=3$ $\dim V_\varepsilon = 4$ $\dim L = 3$

3-manifold

<u>Ex</u>: $n=5$ $\dim L = 7$

Brieskorn: $\{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\}$

$k = 1, \ldots, 28$

The links of these singularities are all homeomorphic to S^7, but pairwise not diffeomorphic.

<u>A_1 singularity</u> $\{z_1^2 + \ldots + z_n^2 = 0\}$

<u>Rmk</u> (Complex) Morse lemma:

If $\det\left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right) \neq 0$ then we can change coordinates to

$f \sim z_1^2 + \ldots + z_n^2$.

$$V_\varepsilon = \{z_1^2 + \ldots + z_n^2 = \varepsilon\} \cap B_\delta^{2n}$$

We can rescale coords and take

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$\varepsilon = 1$, so $\boxed{z_1^2 + \dots + z_n^2 = 1}$

$z_k = x_k + i y_k$ $\quad \begin{cases} \sum x_k^2 - \sum y_k^2 = 1 \\ \sum x_k y_k = 0 \end{cases}$ $\quad \sum x_k^2 = \sum y_k^2 + 1$

Change variables: $\tilde{x}_k = \dfrac{x_k}{\sqrt{1 + \sum y_k^2}}$

$(x_k, y_k) \leftrightarrow (\tilde{x}_k, y_k)$

$$\begin{cases} \sum (\tilde{x}_k)^2 = 1 & \longleftarrow \text{the sphere } S^{n-1} \\ \sum \tilde{x}_k y_k = 0 & \longleftarrow (y_1, \dots, y_n) \text{ is tangent to that sphere} \end{cases}$$

$\{ z_1^2 + \dots + z_n^2 = 1 \} \simeq T^{(*)} S^{n-1}$

total space of tangent bundle to S^{n-1}

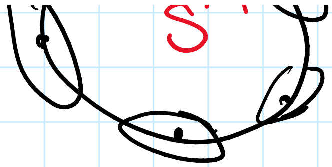
Intersection with B^{2n}_δ

$\sum |z_k|^2 \le \delta$

$\underbrace{\sum x_k^2}_{1 + \sum y_k^2} + \sum y_k^2 \le \delta \quad \Rightarrow \sum y_k^2 \le \dfrac{\delta - 1}{2}$

$\Rightarrow$ disk bundle over S^{n-1}

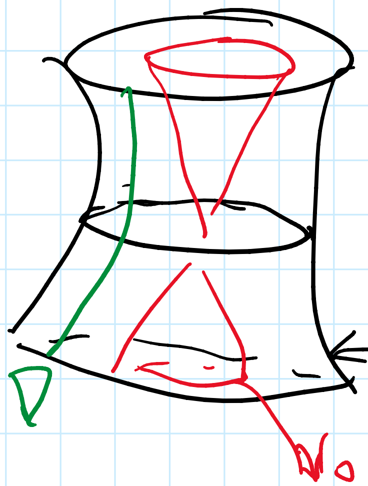
fiber = disk of radius



fiber = disk of radius $\frac{\delta-1}{2}$.

<u>Conclusion</u> The Milnor fiber for A_1 singularity = disk bundle in $T^{(*)}S^{n-1}$.

<u>Ex</u> $\{z_1^2+z_2^2=\varepsilon\}$ = cylinder $= T^{(*)}S^1$



$S^1 = \{z_1^2+z_2^2=\varepsilon\}$

$z_1, z_2 \in \mathbb{R}$

circle of radius $\sqrt{\varepsilon}$, shrinks as $\varepsilon \to 0$

"vanishing cycle"

Can ask various questions (in this ex)

$$H_*(V_\varepsilon) = H_*(S^{n-1}) = \begin{cases} H_0 = \mathbb{Z} \\ H_{n-1} = \mathbb{Z}[\Delta] \end{cases}$$

$$H_*(V_\varepsilon, \partial V_\varepsilon) = \begin{cases} \mathbb{Z}[\nabla] = H_{n-1}(V_\varepsilon, \partial V_\varepsilon) \\ \mathbb{Z} = H_{2n-2}(V_\varepsilon, \partial V_\varepsilon) \end{cases}$$

class of disk in the fiber

Also: V_ε is homotopy equivalent

Also: V_ε is homotopy equivalent to S^{n-1}

$\Rightarrow$ simply connected for $n > 2$.

Q: What is the self-intersection of $\Delta \circ \Delta$?

Δ : $(n-1)$-cycle in $(2n-2)$ mfd

$\Leftrightarrow$ self-intersection of S^{n-1} inside TS^{n-1}

Can deform S^{n-1} using a vector field.

Intersection points $S^{n-1} \cap$ deformed S^{n-1}

= zeroes of the vector field

$$S^{n-1} \circ S^{n-1} = \chi(S^{n-1})$$

$$= \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$\Delta \circ \Delta = \begin{cases} \pm 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

sign depends on n mod 4

$\left(0 \right.$, n even

need to be careful with orientation

$(x_1,y_1,x_2,y_2,\dots x_n,y_n) \longleftrightarrow (x_1,\dots x_n,y_1\dots y_n)$

$\mathbb{C}^n$

$$\Delta \circ \nabla = \pm 1$$

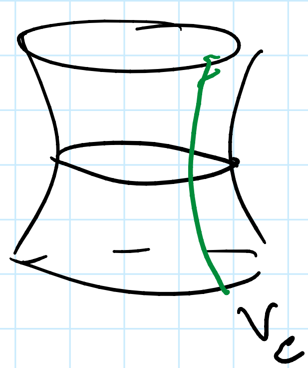
More general questions:

For a general f: · $H_*(V_\varepsilon)$?

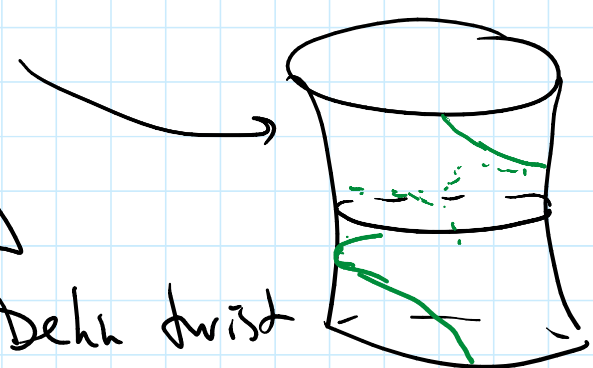
· top. type of V_ε?

· Intersection form?

What happens if we vary ε? X_0



As we go around 0:



$\Delta \longrightarrow \Delta$

$\nabla \longrightarrow$ Dehn twist of ∇ around Δ

$z_1^2+\dots+z_n^2=\varepsilon$

$z_1^2 + \dots + z_n^2 = \varepsilon$

$\tilde{z}_i = \frac{z_i}{\sqrt{\varepsilon}}$