# Topology of Singularities
# Problem set 1

1. (Riemann-Hurwitz formula) Let $C$ be a smooth surface (maybe with boundary) and $\widetilde{C} \to C$ a branched cover of degree $d$. Every point of $C$ has $d$ preimages except for $n$ branch points which have $k_1, \ldots, k_n$ preimages respectively.

(a) Prove that

$$\chi(\widetilde{C}) = d\chi(C) + \sum_{i=1}^{n}(k_i - d)$$

(b) Assume that $C$ and $\widetilde{C}$ have no boundary. Find a relation between the genus of $C$ and the genus of $\widetilde{C}$.

2. (a) Prove that $\{y^2 = x^3 + \varepsilon\}$ is a smooth algebraic curve in $\mathbb{C}^2$ for $\varepsilon \neq 0$.
(b) Use Problem 1 to find its genus by projecting to one of the axis.

3. (a) Prove that the link of singularity $\{x^m = y^n\}$ is the $(m, n)$ torus link with $GCD(m, n)$ components.
(b) Prove that $\{y^m = x^n + \varepsilon\}$ is a smooth algebraic curve in $\mathbb{C}^2$ for $\varepsilon \neq 0$.
(c) Use part (a) and Problem 1 to find the genus of this curve.

4. Consider a smooth algebraic curve $C$ in $\mathbb{CP}^2$ defined by a homogeneous equation $\{f(x, y, z) = 0\}$ of degree $d$. In this problem we will compute its genus as a function of $d$.
(a) Prove that the projection of $C$ to the $x$-axis is a branched cover of degree $d$, with branch points corresponding to vertical tangencies. Assuming that the coefficients of $f$ are generic, prove that each branch point has $(d - 1)$ preimages.
(b) Compute the number of branch points by considering the equations $\{f = 0, \frac{\partial f}{\partial y} = 0\}$ and applying Bezout theorem.
(c) Use (a),(b) and Problem 1 to compute the genus of $C$.