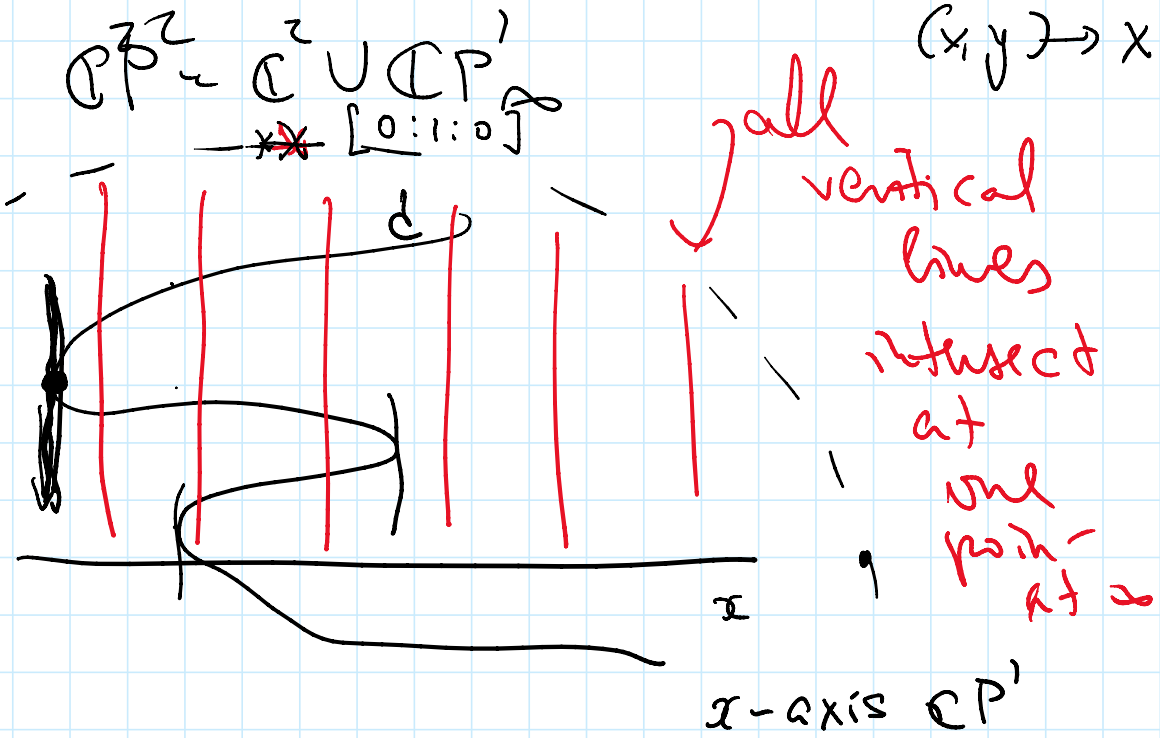
Riemann-Hurwitz

Thursday, January 28, 2021 3:02 PM



Can assume $\infty \in \mathbb{CP}^1$ is not a branch point.

degree d ramified cover $\mathbb{CP}^1$

$\deg \{f(x,y) = 0\} = d$

Vertical tangencies: $\frac{\partial f}{\partial y} = 0$ — $d-1$

$\{f = 0, \frac{\partial f}{\partial y} = 0\}$

Bezout's thm: intersect at $d \cdot (d-1)$ points.

$\Rightarrow d(d-1)$ branch points.

For each of them, $d-1$ preimages.

$$\chi(C) = \chi(\mathbb{CP}^1 - d(d-1)\text{ pts}) \cdot d + \underbrace{d(d-1)}_{\text{pts.}} \cdot (d-1)$$

$$= 2d - d(d-1)d + d(d-1)d - d(d-1)$$

$$= 2d - d^2 + d = 3d - d^2$$

$$= 2 - 2g$$

$$2g = 2 - 3d + d^2 = (d-1)(d-2)$$

$$\boxed{g = \frac{(d-1)(d-2)}{2}}$$

Ex $d=1$ line $g=0$

$d=2$ $g=0$ (quadric)

$d=3$ $g=1$ torus

$d = 3 \qquad g = 1 \qquad$ torus