1. (10 points) Let $x, y, z$ and $w$ be elements if a group $G$.
   a) Solve for $y$ given that $xyz^{-1}w = 1$.
   b) Suppose that $xyz = 1$. Does it follow that $yzx = 1$? Does it follow that $yxz = 1$?

   **Solution:** (a) (5 points) Let us multiply by $x^{-1}$ on the left:
   
   $x^{-1}xyz^{-1}w = x^{-1}, \; yz^{-1}w = x^{-1}$.
   
   Now multiply by $w^{-1}$ on the right:
   
   $yz^{-1}ww^{-1} = yz^{-1} = x^{-1}w^{-1}$,
   
   and multiply by $z$ on the right:
   
   $yz^{-1}z = x^{-1}w^{-1}z$.
   
   Therefore $y = x^{-1}w^{-1}z$.

   (b)(5 points) Let us solve the equation $xyz = 1$ for $z$:
   
   $xyz = 1, \; z = y^{-1}x^{-1}xyz = y^{-1}x^{-1}$.
   
   Now
   
   $yzx = y \cdot y^{-1}x^{-1} \cdot x = 1$.
   
   On the other hand, $yzx \neq 1$ if $xz \neq zx$. For example, pick $x = (1\ 2)$ and $y = (2\ 3)$ in the group $S_3$, then
   
   $z = y^{-1}x^{-1} = (2\ 3)(1\ 2) = (1\ 3\ 2)$,
   
   $yzx = (2\ 3)(1\ 3\ 2)(1\ 2) = e$ while $yzx = (2\ 3)(1\ 2)(1\ 3\ 2) = (1\ 2\ 3) \neq e$.

2. (10 points) How many elements of order 2 does the symmetric group $S_4$ contain?

   **Solution:** The order of a permutations is the least common multiple of the lengths of disjoint cycles in it. If the order is equal to 2, this means that all cycles have length 2. In $S_4$, we have 6 transpositions (cycles of length 2), and 3 products of to disjoint transpositions ($(1\ 2)(3\ 4), (1\ 4)(2\ 3)$ and $(1\ 3)(2\ 4)$). Therefore there are 9 permutations of order 2 in $S_4$.

3. (10 points) Find the order of the permutation
   
   $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 10 & 7 & 6 & 2 & 5 & 9 & 4 & 1 & 8 \end{pmatrix}$

   **Solution:** We have
   
   $f = (1\ 3\ 7\ 9)(2\ 10\ 8\ 4\ 6\ 5)$
   
   The cycles in $f$ have length 4 and 6, so the order of $f$ equals $LCM(4, 6) = 12$.

4. (10 points) Determine if the following subset $H$ is a subgroup of $G$:
   
   a) $G = (\mathbb{R}^*, \cdot)$ and $H = \{-1, 1\}$.
   
   b) $G = (\mathbb{Z}, +)$ and $H$ is the set of positive integers.
   
   c) $G = (\mathbb{R}^*, \cdot)$ and $H$ is the set of positive real numbers.

   **Solution** (a) (4 points) Yes. $H$ contains 1 and is clearly closed under multiplication since $(-1)(-1) = 1$. Also, $(-1)^{-1} = -1$, so $H$ is closed under taking inverses. Therefore $H$ is a subgroup of $G$ (in fact, it is a cyclic subgroup generated by $(-1)$).

   (b) (3 points) No: 1 is in $H$ while $(-1)$ is not in $H$, so $H$ is not closed under taking inverses.
(c) (3 points) Yes: $1 > 0$, so $H$ contains the identity. If $a, b > 0$ then $ab > 0$, so $H$ is closed under multiplication. If $a > 0$ then $a^{-1} = \frac{1}{a} > 0$, so $H$ is closed under taking inverses. Therefore $H$ is a subgroup of $G$. 