1. Solve the system of equations \( x \equiv 3 \mod 5, x \equiv 10 \mod 11 \).

**Solution:** Since \( \text{gcd}(5, 11) = 1 \), by Chinese Remainder Theorem the solution is unique modulo 55. To find this solution, we write all possible elements satisfying the second equation \( x \equiv 10 \mod 11 \):

| \( x \mod 55 \) | 10 | 21 | 32 | 43 | 54 |
| \( x \mod 5 \) | 0  | 1  | 2  | 3  | 4  |

So the solution is \( x = 43 \mod 55 \).

2. Is it possible to construct an injective homomorphism (a) from \( \mathbb{Z}_3 \) to \( \mathbb{Z}_4 \)? (b) From \( S_3 \) to \( S_4 \)?

**Solution:**
(a) No. Suppose \( \varphi : \mathbb{Z}_3 \to \mathbb{Z}_4 \) is a homomorphism, and \( \varphi(1) = a \). Then \( \varphi(2) = \varphi(1 + 1) = \varphi(1) + \varphi(1) = a + a = 2a \). Similarly \( \varphi(3) = 3a \), but 3 = 0 in \( \mathbb{Z}_4 \). So we get the equation \( \varphi(0) = 0 = \varphi(3) = 3a \), so \( 3a = 0 \mod 4 \). Therefore \( a = 0 \mod 4 \), so any homomorphism from \( \mathbb{Z}_3 \) to \( \mathbb{Z}_4 \) sends every element to 0 and is not injective.

(b) Yes, we can extend any permutation \( f \) of three elements to a permutation of 4 elements by \( f(4) = 4 \). Clearly, this is a homomorphism and it is injective.

3. A finite group \( G \) contains an element \( x \) of order 10 and also an element \( y \) of order 6. What can be said about the order of \( G \)?

**Solution:** By Lagrange Theorem \( |G| \) is divisible by 10 and by 6, so it is divisible by \( \text{LCM}(10, 6) = 30 \).

4. Let \( \varphi : G_1 \to G_2 \) be a group homomorphism. Suppose that \( |G_1| = 18, |G_2| = 15 \) and that \( \varphi \) is not the trivial homomorphism. What is the order of the kernel?

**Solution:** By Counting formula we have \( |\text{Ker} \varphi| \cdot |\text{Im} \varphi| = |G_1| = 18 \) and by Lagrange Theorem \( |\text{Im} \varphi| \) divides \( |G_2| = 15 \). Therefore \( |\text{Im} \varphi| \) divides both 15 and 18, so it is either equal to 3 or to 1. Since \( \varphi \) is not trivial, we get \( |\text{Im} \varphi| = 3 \), and by Counting formula \( |\text{Ker} \varphi| = 6 \).