MAT 150A, Fall 2023
Practice problems for the final exam

1. Let $f : S_n \to G$ be any homomorphism (to some group G) such that $f(1\ 2) = e$. Prove that $f(x) = e$ for all x.
2. a) Let x and y be two elements of some group G. Prove that xy and yx are conjugate to each other.
b) Let x and y be two permutations in S_n. Prove that xy and yx have the same cycle type.
3. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$$

and a function $f : G \to \mathbb{R}^*$,

$$f \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a.$$

a) Prove that G is a subgroup of GL_2
b) Prove that f is a homomorphism.
c) Find the kernel and image of f.
4. Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}$$

a) Decompose f into non-intersecting cycles
b) Find the order of f
c) Find the sign of f
d) Compute f^{-1}
5. Find all possible orders of elements in D_6.
6. For every element x of the group D_5:
a) Describe the centralizer of x.
b) Use the Counting Formula to find the size of the conjugacy class of x.
c)* Describe the conujgacy class of x explicitly.
7. Prove that the equation $x^2 + 1 = 4y$ has no integer solutions.
8. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?
9. (a) Prove that any homomorphism from $\mathbb{Z}_{11}$ to S_{10} is trivial.
(b) Find a nontrivial homomorphism from $\mathbb{Z}_{11}$ to S_{11}.
10. Find a nontrivial homomorphism
(a) From S_{11} to $\mathbb{Z}_2$
(b)* From S_{11} to $\mathbb{Z}_4$.
11. How many conjugacy classes are there in S_5?
12. Are the following matrices orthogonal? Do they preserve orientation?

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

13. Prove that for every n there is a group with n elements.
14. Solve the system of equations

$$\begin{cases} x = 1 \mod 8 \\ x = 3 \mod 7. \end{cases}$$

Is the solution unique?
15. Compute $3^{100} \mod 7$.
16. The truncated octahedron (see picture) has 6 square faces and 8 hexagonal faces. Each hexagonal face is adjacent to 3 square and 3 hexagonal faces. Each vertex belongs to two hexagonal and one square face. The group G of isometries acts on vertices, faces and edges.
a) Find the orbit and stabilizer of each face.
b) Use Counting Formula to find the size of G.
c) Find the stabilizer of each vertex and use Counting Formula to find the number of vertices.
d)* There are two types of edges: separating two hexagons, and separating a hexagon from a square. Find the stabilizer of an edge of each type, and use Counting formula to find the number of edges.