MAT 150A, Fall 2023
Practice problems for the final exam

1. Let $f : S_n \to G$ be any homomorphism (to some group $G$) such that $f(1 \ 2) = e$. Prove that $f(x) = e$ for all $x$.

2. a) Let $x$ and $y$ be two elements of some group $G$. Prove that $xy$ and $yx$ are conjugate to each other.
b) Let $x$ and $y$ be two permutations in $S_n$. Prove that $xy$ and $yx$ have the same cycle type.

3. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$$

and a function $f : G \to \mathbb{R}^*$,

$$f \left( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) = a.$$ 

a) Prove that $G$ is a subgroup of $GL_2$
b) Prove that $f$ is a homomorphism.
c) Find the kernel and image of $f$.

4. Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}$$

a) Decompose $f$ into non-intersecting cycles
b) Find the order of $f$
c) Find the sign of $f$
d) Compute $f^{-1}$

5. Find all possible orders of elements in $D_6$.

6. For every element $x$ of the group $D_5$:
a) Describe the centralizer of $x$.
b) Use the Counting Formula to find the size of the conjugacy class of $x$.
c)* Describe the conjugacy class of $x$ explicitly.

7. Prove that the equation $x^2 + 1 = 4y$ has no integer solutions.

8. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?

9. (a) Prove that any homomorphism from $\mathbb{Z}_{11}$ to $S_{10}$ is trivial.
(b) Find a nontrivial homomorphism from $\mathbb{Z}_{11}$ to $S_{11}$.

10. Find a nontrivial homomorphism
(a) From $S_{11}$ to $\mathbb{Z}_2$
(b)* From $S_{11}$ to $\mathbb{Z}_4$

11. How many conjugacy classes are there in $S_5$?

12. Are the following matrices orthogonal? Do they preserve orientation?

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$
13. Prove that for every $n$ there is a group with $n$ elements.

14. Solve the system of equations

$$\begin{cases} 
    x = 1 \pmod{8} \\
    x = 3 \pmod{7}. 
\end{cases}$$

Is the solution unique?

15. Compute $3^{100} \pmod{7}$.

16. The truncated octahedron (see picture) has 6 square faces and 8 hexagonal faces. Each hexagonal face is adjacent to 3 square and 3 hexagonal faces. Each vertex belongs to two hexagonal and one square face. The group $G$ of isometries acts on vertices, faces and edges.

   a) Find the orbit and stabilizer of each face.
   b) Use Counting Formula to find the size of $G$.
   c) Find the stabilizer of each vertex and use Counting Formula to find the number of vertices.
   d)* There are two types of edges: separating two hexagons, and separating a hexagon from a square. Find the stabilizer of an edge of each type, and use Counting formula to find the number of edges.