MAT 150A, Fall 2023
Practice problems for the final exam

1. Let $f : S_n \to G$ be any homomorphism (to some group G) such that $f(1\ 2) = e$. Prove that $f(x) = e$ for all x.

Solution: The kernel of f is a normal subgroup in S_n containing $(1\ 2)$. Since it is normal, it also contains all transpositions. Since it is closed under multiplication and every permutation is a product of transpositions, the kernel coincides with the whole S_n, so $f(x) = e$ for all x.

2. a) Let x and y be two elements of some group G. Prove that xy and yx are conjugate to each other.
b) Let x and y be two permutations in S_n. Prove that xy and yx have the same cycle type.

Solution: a) $yx = y(xy)y^{-1}$.
b) Since xy and yx are conjugate permutations, they have the same cycle type.

3. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$$

and a function $f : G \to \mathbb{R}^*$,

$$f \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a.$$

a) Prove that G is a subgroup of GL_2
b) Prove that f is a homomorphism.
c) Find the kernel and image of f.

Solution: a) We have to check 3 defining properties a subgroup:

- Identity is in G: take $a = 1, b = 0$
- Closed under multiplication:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa_1 & ab_1 + b \\ 0 & 1 \end{pmatrix}$$

- Closed under taking inverses: we need $aa_1 = 1, ab_1 + b = 0$, so

$$a_1 = 1/a,\ b_1 = -b/a.$$

b) From (a) we see that $f(AB) = aa_1 = f(A)f(B)$.
c) Since a can be arbitrary, $Im(f) = \mathbb{R}^*$. Now

$$Ker(f) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \text{ arbitrary} \right\}.$$

4. Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}$$

a) Decompose f into non-intersecting cycles
b) Find the order of f
c) Find the sign of f
d) Compute f^{-1}

Solution: $f = (1\ 5\ 3)(2\ 6)(4\ 7)$, it has order $lcm(3, 2, 2) = 6$ and sign $(-1)^{3-1}(-1)(-1) = 1$, $f^{-1} = (1\ 3\ 5)(2\ 6)(4\ 7)$.

5. Find all possible orders of elements in D_6.

Solution: The group contains reflections and rotations by multiples of $360°/6 = 60°$. Every reflection has order 2, identity has order 1, rotations by $60°$ and $300°$ have order 6, rotations by $120°$ and $240°$ have order 3 and rotation by $180°$ has order 2.

6. For every element x of the group D_5:
a) Describe the centralizer of x.
b) Use the Counting Formula to find the size of the conjugacy class of x.
c)* Describe the conujgacy class of x explicitly.

Solution: a) Clearly, the centralizer of identity is the whole group D_5. If x is a rotation, then all rotations commute with x but no reflection commutes with it, so the centralizer of x consists of all rotations and has 5 elements.

If x is a reflection, then it does not commute with any non-identity rotation, and it does not commute with any reflection y (since xy and yx will be rotations in opposite directions). Therefore for a reflection x the centralizer has two elements $\{1, x\}$.

b) By Counting Formula, we have $|\text{Centralizer}(x)| \cdot |\text{Conj.class}(x)| = |D_5| = 10$. By part (a), the conjugacy class of I has $10/10 = 1$ element, the conjugacy class of a nontrivial rotation consists of $10/5 = 2$ elements, and the conjugacy class of a reflection consists of $10/2 = 5$ elements.

c) If $x = I$ then the conjugacy class is $\{I\}$. If x is a nontrivial rotation then the conjugacy class is $\{x, x^{-1}\}$. If x is a reflection then the conjugacy class consists of all reflections (prove it!).

7. Prove that the equation $x^2 + 1 = 4y$ has no integer solutions.

Solution: Let us consider all possible remainders of x modulo 4. If $x = 0 \mod 4$ then $x^2 + 1 = 1 \mod 4$; if $x = 1 \mod 4$ then $x^2 + 1 = 2 \mod 4$;if $x = 2 \mod 4$ then $x^2 + 1 = 1 \mod 4$;if $x = 3 \mod 4$ then $x^2 + 1 = 2 \mod 4$. Therefore $x^2 + 1$ is never divisible by 4.

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8. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?

Solution: (a) Yes, for example D_3 and $\mathbb{Z}_6$. The orders of elements in D_3 are 1,2,3, while $\mathbb{Z}_6$ has an element of order 6, so they are not isomorphic. (b) No: by Lagrange theorem the order of every element x divides 7, so it should be 1 (and $x = e$) or 7. If the order of x equals 7, then this is just the cyclic group generated by x. So every two groups with 7 elements are cyclic and hence isomorphic. (c) Yes, for example D_4 and $\mathbb{Z}_8$. The orders of elements in D_4 are 1,2,4, while $\mathbb{Z}_8$ has an element of order 8, so they are not isomorphic.

9. (a) Prove that any homomorphism from $\mathbb{Z}_{11}$ to S_{10} is trivial.
(b) Find a nontrivial homomorphism from $\mathbb{Z}_{11}$ to S_{11}.

Solution: (a) Suppose that $f : \mathbb{Z}_{11} \to S_{10}$ is a homomorphism. Then by Counting Formula $|Im(f)|$ divides $|Z_{11}| = 11$. Since 11 is prime, the image of f has either 1 or 11 elements.

On the other hand, by Lagrange Theorem $|Im(f)|$ divides $|S_{10}| = 10!$. Since 11 does not divide 10!, the image must have 1 element, so f is trivial.

(b) We can define $f(k) = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)^k$ for all integer k. Then $f(k + l) = f(k)f(l)$ and $f(11) = e$, so f defines a homomorphism from $\mathbb{Z}_{11}$ to S_{11}.

10. Find a nontrivial homomorphism
(a) From S_{11} to $\mathbb{Z}_2$
(b)* From S_{11} to $\mathbb{Z}_4$.

Solution: (a) The group $(\mathbb{Z}_2, +)$ is isomorphic to $\{\pm 1, \times\}$, so we can send all even permutations to $0 \mod 2$ and all odd permutations to $1 \mod 2$.

(b) We can send all even permutations to $0 \mod 4$ and all odd permutations to $2 \mod 4$. Since $2 + 2 = 0 \mod 4$, this is a homomorphism.

11. How many conjugacy classes are there in S_5?

Solution: The conjugacy classes in S_n correspond to cycle types. There are 7 possible cycle types (listed by length of their cycles): $e, 2, 3, 4, 5, 2 + 2, 2 + 3$.

12. Are the following matrices orthogonal? Do they preserve orientation?

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

Solution: A matrix is orthogonal if $A^t A = I$, and preserves orientation if $\det(A) > 0$, so: (a) Not orthogonal, preserves. (b) Orthogonal, reverses. (c) Orthogonal, preserves.

13. Prove that for every n there is a group with n elements.

Solution: Indeed, consider cyclic group $\mathbb{Z}_n$.

14. Solve the system of equations

$$\begin{cases} x = 1 \mod 8 \\ x = 3 \mod 7. \end{cases}$$

Solution: By Chinese Remainder Theorem, the solution is unique modulo $8 \cdot 7 = 56$. Since $x = 1 \mod 8$, we get

$$x = 1, 9, 17, 25, 33, 41, 49 \mod 56$$

Among these values, only $x = 17$ is equal to 3 modulo 7.

Answer: $x = 17 \mod 56$.

15. Compute $3^{100} \mod 7$.

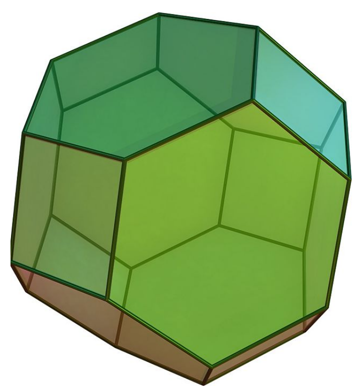
Solution: We have

$$3^1 = 3,\ 3^2 = 2,\ 3^3 = 2{\cdot}3 = 6,\ 3^4 = 6{\cdot}3 = 4,\ 3^5 = 4{\cdot}3 = 5,\ 3^6 = 5{\cdot}3 = 1 \mod 7$$

Therefore

$$3^{100} = (3^6)^{16} \cdot 3^4 = 1^{16} \cdot 4 = 4 \mod 7.$$

16. The truncated octahedron (see picture) has 6 square faces and 8 hexagonal faces. Each hexagonal face is adjacent to 3 square and 3 hexagonal faces. Each vertex belongs to two hexagonal and one square face. The group G of isometries acts on vertices, faces and edges.
a) Find the orbit and stabilizer of each face.
b) Use Counting Formula to find the size of G.
c) Find the stabilizer of each vertex and use Counting Formula to find the number of vertices.
d)* There are two types of edges: separating two hexagons, and separating a hexagon from a square. Find the stabilizer of an edge of each type, and use Counting formula to find the number of edges.



Solution: a) The orbit of a square face consists of all square faces, and the stabilizer of a square face is isomorphic to D_4. The orbit of a hexagonal face consists of all hexagonal faces, and the stabilizer of a hexagonal face is isomorphic to D_3 (it is a subgroup of D_6 which sends the adjacent square faces to square faces).

b) By Counting formula we get $|G| = 6 \cdot |D_4| = 6 \cdot 8 = 48$, and $|G| = 8 \cdot |D_3| = 8 \cdot 6 = 48$ (it is enough to use one of the equations).

c) The stabilizer of a vertex consists of identity and a reflection which swaps two hexagonal faces at this vertex. The orbit of a vertex consists of all vertices. By Counting formula the number of vertices equals $|G|/2 = 48/2 = 24$.

d) Consider an edge separating two hexagons. Its stabilizer has 4 elements: identity, reflection in the plane containing this edge, reflection in a plane perpendicular to this edge, and a rotation by π in a line of interesection of these two planes. By Counting Formula the number of edges of this type equals $48/4 = 12$.

Consider an edge separating a square and a hexagon. Its stabilizer has 2 elements: identity and reflection in the plane containing this edge. By Counting Formula the number of edges of this type equals $48/2 = 24$.

The total number of edges is equal to $12 + 24 = 36$.