1. Let \( f : S_n \to G \) be any homomorphism (to some group \( G \)) such that \( f(1 \ 2) = e \). Prove that \( f(x) = e \) for all \( x \).

Solution: The kernel of \( f \) is a normal subgroup in \( S_n \) containing \((1 \ 2)\). Since it is normal, it also contains all transpositions. Since it is closed under multiplication and every permutation is a product of transpositions, the kernel coincides with the whole \( S_n \), so \( f(x) = e \) for all \( x \).

2. a) Let \( x \) and \( y \) be two elements of some group \( G \). Prove that \( xy \) and \( yx \) are conjugate to each other.

b) Let \( x \) and \( y \) be two permutations in \( S_n \). Prove that \( xy \) and \( yx \) have the same cycle type.

Solution: a) \( yx = y(xy)y^{-1} \).

b) Since \( xy \) and \( yx \) are conjugate permutations, they have the same cycle type.

3. Consider the set

\[
G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}
\]

and a function \( f : G \to \mathbb{R}^* \),

\[
f \left( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) = a.
\]

a) Prove that \( G \) is a subgroup of \( GL_2 \)

b) Prove that \( f \) is a homomorphism.

c) Find the kernel and image of \( f \).

Solution: a) We have to check 3 defining properties a subgroup:

- Identity is in \( G \): take \( a = 1, b = 0 \)

- Closed under multiplication:

\[
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa_1 & ab_1 + b \\ 0 & 1 \end{pmatrix}
\]
• Closed under taking inverses: we need $aa_1 = 1, ab_1 + b = 0$, so
\[ a_1 = 1/a, \ b_1 = -b/a. \]

b) From (a) we see that $f(AB) = aa_1 = f(A)f(B)$.

c) Since $a$ can be arbitrary, $Im(f) = \mathbb{R}^*$. Now
\[ Ker(f) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \text{ arbitrary} \right\}. \]

4. Consider the permutation
\[ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix} \]

a) Decompose $f$ into non-intersecting cycles
b) Find the order of $f$
c) Find the sign of $f$
d) Compute $f^{-1}$

Solution: $f = (1 \ 5 \ 3)(2 \ 6)(4 \ 7)$, it has order $lcm(3, 2, 2) = 6$ and sign $(-1)^{3-1}(-1)(-1) = 1$, $f^{-1} = (1 \ 3 \ 5)(2 \ 6)(4 \ 7)$.

5. Find all possible orders of elements in $D_6$.

Solution: The group contains reflections and rotations by multiples of $360^\circ/6 = 60^\circ$. Every reflection has order 2, identity has order 1, rotations by $60^\circ$ and $300^\circ$ have order 6, rotations by $120^\circ$ and $240^\circ$ have order 3 and rotation by $180^\circ$ has order 2.

6. For every element $x$ of the group $D_5$:
   a) Describe the centralizer of $x$.
   b) Use the Counting Formula to find the size of the conjugacy class of $x$.
   c) Describe the conjugacy class of $x$ explicitly.

Solution: a) Clearly, the centralizer of identity is the whole group $D_5$. If $x$ is a rotation, then all rotations commute with $x$ but no reflection commutes with it, so the centralizer of $x$ consists of all rotations and has 5 elements.

   If $x$ is a reflection, then it does not commute with any non-identity rotation, and it does not commute with any reflection $y$ (since $xy$ and $yx$ will be rotations in opposite directions). Therefore for a reflection $x$ the centralizer has two elements $\{1, x\}$. 

b) By Counting Formula, we have $|\text{Centralizer}(x)| \cdot |\text{Conj.class}(x)| = |D_5| = 10$. By part (a), the conjugacy class of $I$ has $10/10 = 1$ element, the conjugacy class of a nontrivial rotation consists of $10/5 = 2$ elements, and the conjugacy class of a reflection consists of $10/2 = 5$ elements.

c) If $x = I$ then the conjugacy class is $\{I\}$. If $x$ is a nontrivial rotation then the conjugacy class is $\{x, x^{-1}\}$. If $x$ is a reflection then the conjugacy class consists of all reflections (prove it!).

7. Prove that the equation $x^2 + 1 = 4y$ has no integer solutions.

**Solution:** Let us consider all possible remainders of $x$ modulo 4. If $x = 0 \mod 4$ then $x^2 + 1 = 1 \mod 4$; if $x = 1 \mod 4$ then $x^2 + 1 = 2 \mod 4$; if $x = 2 \mod 4$ then $x^2 + 1 = 1 \mod 4$; if $x = 3 \mod 4$ then $x^2 + 1 = 2 \mod 4$. Therefore $x^2 + 1$ is never divisible by 4.

8. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?

**Solution:** (a) Yes, for example $D_3$ and $Z_6$. The orders of elements in $D_3$ are 1,2,3, while $Z_6$ has an element of order 6, so they are not isomorphic. (b) No: by Lagrange theorem the order of every element $x$ divides 7, so it should be 1 (and $x = e$) or 7. If the order of $x$ equals 7, then this is just the cyclic group generated by $x$. So every two groups with 7 elements are cyclic and hence isomorphic. (c) Yes, for example $D_4$ and $Z_8$. The orders of elements in $D_4$ are 1,2,4, while $Z_8$ has an element of order 8, so they are not isomorphic.

9. (a) Prove that any homomorphism from $Z_{11}$ to $S_{10}$ is trivial.
(b) Find a nontrivial homomorphism from $Z_{11}$ to $S_{11}$.

**Solution:** (a) Suppose that $f : Z_{11} \rightarrow S_{10}$ is a homomorphism. Then by Counting Formula $|\text{Im}(f)|$ divides $|Z_{11}| = 11$. Since 11 is prime, the image of $f$ has either 1 or 11 elements.

On the other hand, by Lagrange Theorem $|\text{Im}(f)|$ divides $|S_{10}| = 10!$. Since 11 does not divide 10!, the image must have 1 element, so $f$ is trivial.

(b) We can define $f(k) = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11)^k$ for all integer $k$. Then $f(k + l) = f(k)f(l)$ and $f(11) = e$, so $f$ defines a homomorphism from $Z_{11}$ to $S_{11}$.

10. Find a nontrivial homomorphism
(a) From $S_{11}$ to $Z_2$
(b)* From $S_{11}$ to $Z_4$. 

3
Solution: (a) The group \((\mathbb{Z}_2, +)\) is isomorphic to \(\{\pm 1, \times\}\), so we can send all even permutations to 0 mod 2 and all odd permutations to 1 mod 2.

(b) We can send all even permutations to 0 mod 4 and all odd permutations to 2 mod 4. Since \(2 + 2 = 0\) mod 4, this is a homomorphism.

11. How many conjugacy classes are there in \(S_5\)?

Solution: The conjugacy classes in \(S_n\) correspond to cycle types. There are 7 possible cycle types (listed by length of their cycles): \(e, 2, 3, 4, 5, 2+2, 2+3\).

12. Are the following matrices orthogonal? Do they preserve orientation?

\[
\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}.
\]

Solution: A matrix is orthogonal if \(A^tA = I\), and preserves orientation if \(\det(A) > 0\), so: (a) Not orthogonal, preserves. (b) Orthogonal, reverses. (c) Orthogonal, preserves.

13. Prove that for every \(n\) there is a group with \(n\) elements.

Solution: Indeed, consider cyclic group \(\mathbb{Z}_n\).

14. Solve the system of equations

\[
\begin{align*}
x & = 1 \mod 8 \\
x & = 3 \mod 7.
\end{align*}
\]

Solution: By Chinese Remainder Theorem, the solution is unique modulo \(8 \cdot 7 = 56\). Since \(x = 1 \mod 8\), we get

\[x = 1, 9, 17, 25, 33, 41, 49 \mod 56\]

Among these values, only \(x = 17\) is equal to 3 modulo 7.

Answer: \(x = 17 \mod 56\).

15. Compute \(3^{100} \mod 7\).

Solution: We have

\[3^1 = 3, \ 3^2 = 2, \ 3^3 = 2 \cdot 3 = 6, \ 3^4 = 6 \cdot 3 = 4, \ 3^5 = 4 \cdot 3 = 5, \ 3^6 = 5 \cdot 3 = 1 \mod 7\]
Therefore

\[ 3^{100} = (3^6)^{16} \cdot 3^4 = 1^{16} \cdot 4 = 4 \mod 7. \]

16. The truncated octahedron (see picture) has 6 square faces and 8 hexagonal faces. Each hexagonal face is adjacent to 3 square and 3 hexagonal faces. Each vertex belongs to two hexagonal and one square face. The group \( G \) of isometries acts on vertices, faces and edges.

a) Find the orbit and stabilizer of each face.

b) Use Counting Formula to find the size of \( G \).

c) Find the stabilizer of each vertex and use Counting Formula to find the number of vertices.

d)* There are two types of edges: separating two hexagons, and separating a hexagon from a square. Find the stabilizer of an edge of each type, and use Counting formula to find the number of edges.

Solution: a) The orbit of a square face consists of all square faces, and the stabilizer of a square face is isomorphic to \( D_4 \). The orbit of a hexagonal face consists of all hexagonal faces, and the stabilizer of a hexagonal face is isomorphic to \( D_3 \) (it is a subgroup of \( D_6 \) which sends the adjacent square faces to square faces).

b) By Counting formula we get \(|G| = 6 \cdot |D_4| = 6 \cdot 8 = 48\), and \(|G| = 8 \cdot |D_3| = 8 \cdot 6 = 48\) (it is enough to use one of the equations).

c) The stabilizer of a vertex consists of identity and a reflection which swaps two hexagonal faces at this vertex. The orbit of a vertex consists of all vertices. By Counting formula the number of vertices equals \(|G|/2 = 48/2 = 24\).

d) Consider an edge separating two hexagons. Its stabilizer has 4 elements: identity, reflection in the plane containing this edge, reflection in a
plane perpendicular to this edge, and a rotation by $\pi$ in a line of intersection of these two planes. By Counting Formula the number of edges of this type equals $48/4 = 12$.

Consider an edge separating a square and a hexagon. Its stabilizer has 2 elements: identity and reflection in the plane containing this edge. By Counting Formula the number of edges of this type equals $48/2 = 24$.

The total number of edges is equal to $12 + 24 = 36$. 