MAT 150A, Fall 2023
Solutions to Homework 1

1. (25 points) Compute the products fg and gf for the permutations

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 7 & 1 & 2 & 6 & 5 \end{pmatrix},\ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Solution: We have

$$fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 2 & 1 & 7 & 3 & 4 \end{pmatrix},\ gf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 1 & 7 & 6 & 2 & 3 \end{pmatrix}$$

2. (25 points) Write the following permutations as products of disjoint cycles:

$$(1\ 2)(1\ 3)(1\ 4)(1\ 5),\ (1\ 2\ 3)(2\ 3\ 4)(3\ 4\ 5),\ (1\ 2\ 3\ 4)(2\ 3\ 4\ 5),\ (1\ 2)(2\ 3)(3\ 4)(4\ 5)(5\ 1).$$

Solution:

$$(1\ 2)(1\ 3)(1\ 4)(1\ 5) = (1\ 5\ 4\ 3\ 2),\ (1\ 2\ 3)(2\ 3\ 4)(3\ 4\ 5) = (1\ 2)(4\ 5),$$
$$(1\ 2\ 3\ 4)(2\ 3\ 4\ 5) = (1\ 2\ 4\ 5\ 3),$$
$$(1\ 2)(2\ 3)(3\ 4)(4\ 5)(5\ 1) = (2\ 3\ 4\ 5).$$

3. (25 points) Present the following permutation as a product of disjoint cycles for all k:

$$(1\ 2\ 3)^k = \underbrace{(1\ 2\ 3)\cdots(1\ 2\ 3)}_{k \text{ times}}.$$

Solution: We have $(1\ 2\ 3)^2 = (1\ 3\ 2)$ and $(1\ 2\ 3)^3 = (1\ 2\ 3)^2(1\ 2\ 3) = (1\ 3\ 2)(1\ 2\ 3) = e$. Therefore

$$(1\ 2\ 3)^4 = (1\ 2\ 3)^3(1\ 2\ 3) = e \cdot (1\ 2\ 3) = (1\ 2\ 3),$$
$$(1\ 2\ 3)^5 = (1\ 2\ 3)^3(1\ 2\ 3)^2 = e \cdot (1\ 2\ 3)^2 = (1\ 3\ 2).$$

In general, the powers of $(1\ 2\ 3)^k$ will repeat with period 3 as follows:

$$(1\ 2\ 3)^k = \begin{cases} ((1\ 2\ 3)^3)^m = e^m = e, & k = 3m \\ (1\ 2\ 3)^{3m}(1\ 2\ 3) = e(1\ 2\ 3) = (1\ 2\ 3), & k = 3m+1 \\ (1\ 2\ 3)^{3m}(1\ 2\ 3)^2 = e(1\ 2\ 3)^2 = (1\ 3\ 2), & k = 3m+2. \end{cases}$$

4. (25 points) Find the number of even and odd permutations in S_n for $n = 1, 2, 3, 4$.

Solution 1: We can just count all permutations directly. For $n = 1$ there is only identity permutation and it is even. For $n = 2$ we have $S_2 = \{e, (1\ 2)\}$, and e is even while $(1\ 2)$ is odd. For $n = 3$ we have 3 even permutations $\{e, (1\ 2\ 3), (1\ 3\ 2)\}$, and 3 odd permutations $\{(12), (13), (23)\}$. For $n = 4$ we have 12 even permutations:

$$\{e,\ 8 \text{ cycles of length } 3,\ (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

as well as 12 odd permutations: 6 transpositions and 6 cycles of length 4.

Solution 2: Suppose $n \geq 2$. If f is an even permutation, define $T(f) = f \cdot (1\ 2)$. Since

$$\text{sgn}(T(f)) = \text{sgn}(f) \cdot \text{sgn}(1\ 2) = -\text{sgn}(f),$$

the permutations f and $T(f)$ have opposite signs. Furthermore,

$$T(T(f)) = f \cdot (1\ 2) \cdot (1\ 2) = f,$$

so the function T on the set of permutations is inverse to itself and hence a bijection. Therefore T is a bijection from the set of even permutation to the set of odd permutations and these two sets have the same number of elements. Since the total number of permutations equals $n!$, both the number of even permutations and the number of odd ones are equal to $n!/2$.