Solutions to Homework 1

1. (25 points) Compute the products \( fg \) and \( gf \) for the permutations
\[
f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 7 & 1 & 2 & 6 & 5 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.
\]

**Solution:** We have
\[
fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 2 & 1 & 7 & 3 & 4 \end{pmatrix}, \quad gf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 1 & 7 & 6 & 2 & 3 \end{pmatrix}.
\]

2. (25 points) Write the following permutations as products of disjoint cycles:
\[(1 2)(1 3)(1 4)(1 5), (1 2 3)(2 3 4)(3 4 5), (1 2 3 4)(2 3 4 5), (1 2)(2 3)(3 4)(4 5)(5 1).\]

**Solution:**
\[(1 2)(1 3)(1 4)(1 5) = (1 5 4 3 2), (1 2 3)(2 3 4)(3 4 5) = (1 2)(4 5), (1 2 3 4)(2 3 4 5) = (1 2 4 5 3), (1 2)(2 3)(3 4)(4 5)(5 1) = (2 3 4 5).\]

3. (25 points) Present the following permutation as a product of disjoint cycles for all \( k \):
\[(1 2 3)^k = \underbrace{(1 2 3) \cdots (1 2 3)}_{k \text{ times}}.
\]

**Solution:** We have \((1 2 3)^2 = (1 3 2)\) and \((1 2 3)^3 = (1 2 3)^2(1 2 3) = (1 3 2)(1 2 3) = e.\)
Therefore
\[(1 2 3)^4 = (1 2 3)^3(1 2 3) = e \cdot (1 2 3) = (1 2 3), \quad (1 2 3)^5 = (1 2 3)^3(1 2 3)^2 = e \cdot (1 2 3)^2 = (1 3 2).
\]
In general, the powers of \((1 2 3)^k\) will repeat with period 3 as follows:
\[
(1 2 3)^k = \begin{cases} 
((1 2 3)^3)^m = e^m = e, & k = 3m \\
(1 2 3)^{3m}(1 2 3) = e(1 2 3) = (1 2 3), & k = 3m + 1 \\
(1 2 3)^{3m}(1 2 3)^2 = e(1 2 3)^2 = (1 3 2), & k = 3m + 2.
\end{cases}
\]

4. (25 points) Find the number of even and odd permutations in \( S_n \) for \( n = 1, 2, 3, 4, \)

**Solution 1:** We can just count all permutations directly. For \( n = 1 \) there is only identity permutation and it is even. For \( n = 2 \) we have \( S_2 = \{ e, (1 2) \} \), and \( e \) is even while \((1 2)\) is odd. For \( n = 3 \) we have 3 even permutations \( \{ e, (1 2 3), (1 3 2) \} \), and 3 odd permutations \( \{(12), (13), (23)\} \). For \( n = 4 \) we have 12 even permutations:
\[
\{ e, \text{ 8 cycles of length 3}, \ (1 2)(3 4), \ (1 3)(2 4), \ (1 4)(2 3) \}
\]
as well as 12 odd permutations: 6 transpositions and 6 cycles of length 4.

**Solution 2:** Suppose \( n \geq 2 \). If \( f \) is an even permutation, define \( T(f) = f \cdot (1 2) \). Since
\[
\text{sgn}(T(f)) = \text{sgn}(f) \cdot \text{sgn}(1 2) = -\text{sgn}(f),
\]
the permutations \( f \) and \( T(f) \) have opposite signs. Furthermore,
\[
T(T(f)) = f \cdot (1 2) \cdot (1 2) = f;
\]
so the function $T$ on the set of permutations is inverse to itself and hence a bijection. Therefore $T$ is a bijection from the set of even permutation to the set of odd permutations and these two sets have the same number of elements. Since the total number of permutations equals $n!$, both the number of even permutations and the number of odd ones are equal to $n!/2$. 