1. (25 points) Let $G$ be a group of order 25. Prove that $G$ has at least one subgroup of order 5, and that if it contains only one subgroup of order 5 then it is a cyclic group.

**Solution:** Let $x$ be an element of $G$. By Lagrange Theorem the order of $x$ divides 25, so it could be equal to 1, 5 or 25. We have the following cases:

a) $\text{Ord}(x) = 1$, then $x = e$ is the identity element.

b) $\text{Ord}(x) = 25$, then $G$ is cyclic and generated by $x$. It has a subgroup $\langle x^5 \rangle = \{1, x^5, x^{10}, x^{15}, x^{20}\}$ of order 5.

c) Order of every non-identity element equals 5. Then each non-identity element generates a subgroup of order 5. If $\{1, x, x^2, x^3, x^4\}$ is one such subgroup and $y$ is not contained in it then $y$ generates another subgroup of order 5. So we have more than one subgroup of order 5.

2. (25 points) Is it possible to construct an injective homomorphism (a) from $\mathbb{Z}_3$ to $\mathbb{Z}_4$? (b) From $S_3$ to $S_4$?

**Solution:** (a) (15 points) **(First solution)** No. Assume that $\varphi : \mathbb{Z}_3 \to \mathbb{Z}_4$ is an injective homomorphism, then $\text{Im}(\varphi)$ is a subgroup $\mathbb{Z}_4$ with 3 elements. By Lagrange Theorem the size of $\text{Im}(\varphi)$ must divide $|\mathbb{Z}_4| = 4$, contradiction.

**(Second solution)** Assume that $\varphi(1) = a$, then $\varphi(3) = \varphi(1 + 1 + 1) = \varphi(1) + \varphi(1) + \varphi(1) = a + a + a = 3a$. But $\varphi(3) = \varphi(0) = 0$, so $3a = 0$ in $\mathbb{Z}_4$. Since 3 and 4 are coprime, we get $a = 0 \mod 4$, contradiction.

(b) (10 points) Yes: given a permutation $f$ in $S_3$, the permutation $\varphi(f)$ in $S_4$ is defined by

$$\varphi(f)(i) = \begin{cases} f(i) & i = 1, 2, 3 \\ 4 & i = 4. \end{cases}$$

so that $\varphi(f)$ fixes 4. Clearly, this is an injective homomorphism.

3. (25 points) A finite group $G$ contains an element $x$ of order 10 and also an element $y$ of order 6. What can be said about the order of $G$?

**Solution:** By Lagrange Theorem the order of $G$ is divisible by 10 and by 6, so it is divisible by 30.

4. (25 points) Let $\varphi : G_1 \to G_2$ be a group homomorphism. Suppose that $|G_1| = 18, |G_2| = 15$ and that $\varphi$ is not the trivial homomorphism. What is the order of the kernel?

**Solution:** By Counting Formula $|\text{Ker}\varphi| \cdot |\text{Im}\varphi| = |G_1| = 18$ and by Lagrange Theorem $|\text{Im}\varphi|$ divides $|G_2| = 15$. Therefore $|\text{Im}\varphi|$ divides both 15 and 18 and it could be equal to 1 or 3. Since $\varphi$ is nontrivial, we get $|\text{Im}\varphi| = 3$ and $|\text{Ker}\varphi| = 6$. 
