MAT 150A, Fall 2023
Solutions to homework 5

The first three problems are focused on the set U of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where x is an arbitrary real number.

1. (25 points) Prove that U is a subgroup of $GL(2)$.

Solution: We check three axioms of a subgroup:

1) Closure:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$$

2) Identity: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $x = 0$.

3) Inverse: $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ is contained in U.

2. (25 points) Prove that the group U is isomorphic to $(\mathbb{R}, +)$.

Solution: We construct a function $\varphi : \mathbb{R} \to U$ by the equation $\varphi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Clearly, it is a bijection. Furthermore,

$$\varphi(x)\varphi(y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = \varphi(x+y),$$

so φ is a homomorphism. Since it is a bijection and a homomorphisms, it is an isomorphism.

3. (25 points) Is U a **normal** subgroup of $GL(2)$? Please explain your answer in detail.

Solution: No. Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then $X^{-1} = X$ and

$$XAX^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

If $x \neq 0$, this is not in U, so XAX^{-1} is not in U. Therefore U is not normal.

4. (25 points) Use Chinese Remainder Theorem to solve the system of equations

$$x = 3 \mod 5, \; x = 4 \mod 11.$$

Solution: By Chinese Reminder Theorem, since $GCD(5, 11) = 1$, the solution is unique $\mod 55$. Now we can use the first equation $x = 3 \mod 5$ and write all possible $x \mod 55$ in the table:

$x \mod 55$	3	8	13	18	23	28	33	38	43	48	53
$x \mod 11$	3	8	2	7	1	6	0	5	10	4	9

Therefore $x = 48 \mod 55$.