MAT 150A, Fall 2023
Solutions to homework 6

1. (25 points) Are the following matrices orthogonal? If they are, describe them geometrically.

   (a) (15 points) \[
   \begin{pmatrix}
   1 & 1 \\
   1 & 1 \\
   \end{pmatrix}
   \]

   Solution: a) We have
   \[
   A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq I
   \]
   so \( A \) is not orthogonal.

   (b) (10 points) \[
   \begin{pmatrix}
   \sqrt{3}/2 & 1/2 \\
   1 & -\sqrt{3}/2 \\
   \end{pmatrix}
   \]

   Solution: b) We have
   \[
   B^T B = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1 & -\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1 & -\sqrt{3}/2 \end{pmatrix} =
   \begin{pmatrix} 3/4 + 1/4 & \sqrt{3}/4 - \sqrt{3}/4 \\ \sqrt{3}/4 - \sqrt{3}/4 & 3/4 + 3/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I
   \]
   so \( B \) is orthogonal.

2. (25 points) Find all diagonal \( 3 \times 3 \) matrices
   \[
   A = \begin{pmatrix}
   a & 0 & 0 \\
   0 & b & 0 \\
   0 & 0 & c \\
   \end{pmatrix}
   \]
   which are orthogonal.

   Solution: We have
   \[
   A^T A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}
   \]
   The matrix \( A \) is orthogonal if \( a^2 = b^2 = c^2 = 1 \), so \( a, b, c \in \{-1, 1\} \).

   Recall that an orthogonal matrix \( A \) is called \textbf{orientation reversing} if \( \det(A) = -1 \) and \textbf{orientation preserving} if \( \det(A) = 1 \).

3. (25 points) Find all \( 2 \times 2 \) orientation reversing matrices of finite order.

   Solution: Any orientation reversing \( 2 \times 2 \) orthogonal matrix is a reflection and has order 2.

4. (25 points) Find all \( 2 \times 2 \) orientation preserving matrices of finite order.
Solution: Any orientation reversing $2 \times 2$ orthogonal matrix is a rotation by some order $\varphi$. It has finite order $n$ if $m\varphi = 2\pi n$ and $\varphi = \frac{2\pi n}{m}$ for some integer $n$. 