1. Recall that any isometry of the plane can be written as

\[ m(x) = Ax + b \]

where \( A \) is an orthogonal matrix and \( b \) is a fixed vector. Assume that \( \det(A) = -1 \), so \( m \) reverses orientation. Prove that \( m^2 \) is a translation by some vector.

**Solution 1:** Since \( A \) is an orthogonal \( 2 \times 2 \) matrix and \( \det A = -1 \), by classification theorem \( A \) is a reflection, therefore \( A^2 = I \). Therefore \( m(m(x)) = A(Ax + b) + b = A^2x + Ab + b = x + Ab + b \), so \( m^2(x) \) is a translation by the vector \( Ab + b \).

**Solution 2:** By classification theorem \( m \) is either a reflection or a glide reflection. If \( m \) is a reflection then \( m^2 \) is identity. If \( m \) is a glide reflection, it is a composition of reflection in some line \( \ell \), and translation by a vector \( b \) parallel to \( \ell \). Then \( m^2 \) is a translation by \( 2b \).

In problems 2-4, find all isometries of the following infinite patterns:

2. \( \ldots \perp \perp \perp \perp \perp \perp \perp \ldots \)

**Solution:** Assume that all \( \perp \) are placed at the points \((n, 0)\) for integer \( n \). Then we have the following isometries:

- Translation by \((m, 0)\) for any integer \( m \)
- Reflection in the vertical line \( x = n \) for integer \( n \)
- Reflection in the vertical line \( x = n + \frac{1}{2} \) for integer \( n \) (between two \( \perp \))

3. \( \ldots \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \ldots \)

**Solution:** Assume that all \( \leftarrow \) are placed at the points \((n, 0)\) for integer \( n \). Then we have the following isometries:

- Translation by \((m, 0)\) for any integer \( m \)
- Reflection in the horizontal line \( y = 0 \)
- Glide reflection: reflection in the horizontal line followed by a translation by \((m, 0)\) for any integer \( m \).

4. \( \ldots \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \leftarrow \ldots \)

**Solution:** Assume that all \( \leftarrow \) are placed at the points \((2n, 0)\) and \( \rightarrow \) are placed at the points \((2n + 1, 0)\) for integer \( n \). Then we have the following isometries:

- Translation by \((2m, 0)\) for any integer \( m \)
- Reflection in the horizontal line \( y = 0 \)
- Glide reflection: reflection in the horizontal line followed by a translation by \((m, 0)\) for any integer \( m \).
- Reflection in the vertical line \( x = n + \frac{1}{2} \) for all integer \( n \) (between \( \leftarrow \rightarrow \) or between \( \rightarrow \leftarrow \))
- Rotation by \( \pi \) around the point \((n + \frac{1}{2}, 0)\).

Note that rotation by \( \pi \) can be obtained as a composition of reflections in the vertical and in the horizontal lines in either order.