MAT 150A, Fall 2023
Solutions to homework 7

1. Recall that any isometry of the plane can be written as

$$m(x) = Ax + b$$

where A is an orthogonal matrix and b is a fixed vector. Assume that $\det(A) = -1$, so m reverses orientation. Prove that m^2 is a translation by some vector.

Solution 1: Since A is an orthogonal 2×2 matrix and $\det A = -1$, by classification theorem A is a reflection, therefore $A^2 = I$. Therefore $m(m(x)) = A(Ax+b)+b = A^2x + Ab + b = x + Ab + b$, so $m^2(x)$ is a translation by the vector $Ab + b$.

Solution 2: By classification theorem m is either a reflection or a glide reflection. If m is a reflection then m^2 is identity. If m is a glide reflection, it is a composition of reflection in some line ℓ, and translation by a vector b parallel to ℓ. Then m^2 is a translation by $2b$.

In problems 2-4, find all isometries of the following infinite patterns:

2. $\ldots \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \ldots$

Solution: Assume that all $\perp$ are placed at the points $(n, 0)$ for integer n. Then we have the following isometries:

- Translation by $(m, 0)$ for any integer m
- Reflection in the vertical line $x = n$ for integer n
- Reflection in the vertical line $x = n + \frac{1}{2}$ for integer n (between two $\perp$)

3. $\ldots \lhd \quad \lhd \quad \lhd \quad \lhd \quad \lhd \quad \lhd \quad \lhd \quad \ldots$

Solution: Assume that all $\lhd$ are placed at the points $(n, 0)$ for integer n. Then we have the following isometries:

- Translation by $(m, 0)$ for any integer m
- Reflection in the horizontal line $y = 0$
- Glide reflection: reflection in the horizontal line followed by a translation by $(m, 0)$ for any integer m.

4. $\ldots \lhd \quad \rhd \quad \lhd \quad \rhd \quad \lhd \quad \rhd \quad \lhd \quad \ldots$

Solution: Assume that all $\lhd$ are placed at the points $(2n, 0)$ and $\rhd$ are placed at the points $(2n+1, 0)$ for integer n. Then we have the following isometries:

- Translation by $(2m, 0)$ for any integer m
- Reflection in the horizontal line $y = 0$
- Glide reflection: reflection in the horizontal line followed by a translation by $(m, 0)$ for any integer m.
- Reflection in the vertical line $x = n + \frac{1}{2}$ for all integer n (between $\lhd \quad \rhd$ or between $\rhd \quad \lhd$)
- Rotation by π around the point $(n + \frac{1}{2}, 0)$.

Note that rotation by π can be obtained as a composition of reflections in the vertical and in the horizontal lines in either order.