MAT 150A, Fall 2023
Solutions to homework 8

1. (20 points) A brick has height 1, width 2 and length 3. Its group of isometries acts on faces, vertices and edges of the brick.
a) Find the orbit and stabilizer for each face of the brick.
b) Find the orbit and stabilizer for each vertex of the brick.
c) Find the orbit and stabilizer for each edge of the brick.
d) Use Counting Formula to find the size of the isometry group.

Solution: (a) Let us choose a system of coordinates such that the origin is at the center of the brick, the x-axis is parallel to the edge of length 1, the y-axis is parallel to the edge of length 2, and the z-axis is parallel to the edge of length 3.

Now the orbit of a 1×2 face consists of this face and its opposite 1×2 face, while its stabilizer has 4 elements: I, reflection in the xz-plane, reflection in the yz-plane and the rotation by π around the z-axis (perpendicular to the face). In fact, the stabilier is isomorphic the isometry group of the 1×2 rectangle in the plane, extended in z-direction. The orbits and stabilizers of other faces are similar.

(b) The orbit of a vertex consists of all 8 vertices of the brick, since one can get from any vertex to any other vertex by a sequence of reflections. The stabilizer of a vertex is trivial.

(c) The orbit of an edge consists of 4 edges of the same length, so that the orbit of a length 1 edge consists of all length 1 edges. The stabilizer of an edge has two elements: I and reflection in the plane through the middle of the edge and perpendicular to it.

(d) By Counting Formula we have $|G| = 2 \times 4 = 8 \times 1 = 4 \times 2 = 8$.

2. (10 points) Consider the action of the group S_7 on the set of 3-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$.
a) Find the stabilizer of the subset $\{1, 2, 3\}$.
b) Describe the orbit of $\{1, 2, 3\}$. and use Counting Formula to compute the size of this orbit.

Solution: (a) We have $g\{1, 2, 3\} = \{g(1), g(2), g(3)\}$, therefore the stabilizer of $\{1, 2, 3\}$ consists of permutations g which permute 1,2, 3 in some order. Such a permutation is also permuting 4,5,6,7 in some order, and so the stabilizer is isomorphic to $S_3 \times S_4$ and has $3! \cdot 4!$ elements.

(b) The orbit of $\{1, 2, 3\}$ consists of all 3-element subsets. By Counting Formula the number of such subsets equals

$$\frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3!} = 35.$$

3. (10 points) A soccer ball (see picture[1]) has 12 black pentagons and 20 white hexagons. Its group of isometries acts on the set of faces.
a) Find the orbit and stabilizer of each face
b) Use Counting Formula to compute the size of the isometry group.

Solution: (a) The orbit of each pentagon face consists of all pentagon faces, the orbit of each hexagon face consists of all hexagon faces.

The stabilizer of a pentagon face is isomorphic to the dihedral group D_5. The stabilizer of the hexagon face is isomorphic to the subgroup of D_6 sending three black neighboring pentagons (see picture above) to black pentagons, so it is in fact isomorphic to D_3.

(b) By Counting Formula, we get $|G| = 12 \cdot |D_5| = 12 \cdot 10 = 120$ and $|G| = 20 \cdot |D_3| = 20 \cdot 6 = 120$.

[1]Picture credit: Wikipedia