MAT 150C, Spring 2021
Solutions to Homework 1

1. Consider the cyclic group $G_n = \langle x | x^n = 1 \rangle$.
a) (15 points) Describe all one-dimensional complex representations of G_n.
b) (15 points) Prove that every complex representation of G_n has a one-dimensional invariant subspace.

Solution: a) In a one-dimensional representation the image of x is a 1×1 invertible complex matrix, that is, a nonzero complex number a. Since $x^n = 1$, we get $a^n = 1$, so

$$a = e^{\frac{2\pi i k}{n}} = \cos(\frac{2\pi k}{n}) + i \sin(\frac{2\pi k}{n})$$

is a n-th root of unity. Here k is an arbitrary integer, but we can choose $k = 0, 1, \ldots, n-1$.

Given such a, we have $\rho(1) = 1, \rho(x) = a, \rho(x^2) = a^2, \ldots, \rho(x^{n-1}) = a^{n-1}$. Since $a^n = 1$, this is a well-defined homomorphism from G to $GL(1)$.

b) Let $\rho : G \to GL(V)$ be a representation. Let v be an eigenvector for $\rho(x)$ with eigenvalue λ. Then

$$\rho(x)(v) = \lambda v,\ \rho(x^2)(v) = \rho(x)(\rho(x)(v)) = \rho(x)(\lambda v) = \lambda^2 v$$

and so on, hence $\rho(x^k)(v) = \lambda^k v$. Therefore v is a common eigenvector for all matrices $\rho(x^k)$, and spans a one-dimensional invariant subspace.

2. a) (15 points) Prove that there is a two-dimensional representation of G_4 such that

$$x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

b) (15 points) Find all invariant subspaces for the corresponding **real** representation.
c) (15 points) Find all invariant subspaces for the corresponding **complex** representation.

Solution: a) We have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = I.$$

Therefore $\rho(x)^4 = I$ and this is a well defined representation of the cyclic group of order 4.

b)c) Let us find the eigenvectors and eigenvalues for $\rho(x)$. The characteristic polynomial equals

$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1,$$

so the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. It is easy to see that the eigenvectors are $(1, i)$ and $(1, -i)$.

Since the eigenvectors are not real, the only real invariant subspaces in (b) are 0 and $\mathbb{R}^2$.

In (c), there are 4 invariant subspaces: $0, \mathbb{C}^2, \mathrm{span}(1, i), \mathrm{span}(1, -i)$.

3. (25 points) Consider the standard two-dimensional representation of the dihedral group D_n. For which n is this an irreducible **complex** representation.

Solution: Assume that this representation is not irreducible, then it has a 1-dimensional invariant subspace or, equivalently, a common eigenvector for all operators in D_n.

Observe that for a reflection in a line ℓ there are two eigenvectors: a vector along ℓ has eigenvalue 1, and a vector perpendicular to ℓ has eigevalue (-1). If ℓ_1 and ℓ_2 are two distinct lines which are not perpendicular, then the corresponding reflections not have common eigenvectors and hence the representation is irreducible. For any $n \geq 3$ we can find such lines.

Alternatively, one can argue that the rotations have complex eigenvectors and reflections have real eigenvectors, so they do not have common eigenvectors as well.