MAT 150C, Spring 2021
Solutions to Homework 2

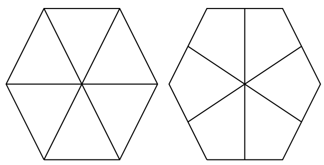
1. a) Prove that for odd n all reflections in the dihedral group D_n are conjugate to each other.
b) Prove that for even n there are exactly two conjugacy classes of reflections in D_n.

Solution: The composition of two reflections separated by angle φ is a rotation by angle 2φ. Suppose that ℓ_1, ℓ_2 and ℓ_3 are three lines such that the angles between ℓ_1 and ℓ_2, and between ℓ_2 and ℓ_3 are equal to φ. Then the corresponding reflections satisfy the equation

$$R_{\ell_1}R_{\ell_2} = R_{\ell_2}R_{\ell_3},\ R_{\ell_2}^{-1}R_{\ell_1}R_{\ell_2} = R_{\ell_3}.$$

Therefore the reflections in ℓ_1 and ℓ_2 are conjugate to each other, and the angle between ℓ_1 and ℓ_3 equals 2φ.

The minimal angle between two lines of reflection in the n-gon equals $\frac{2\pi}{2n}$, and by the above any two reflections separated by an angle $2k\cdot\frac{2\pi}{2n} = \frac{2\pi k}{n}$ are conjugate to each other. If n is odd, this means that all reflections are conjugate. If n is even, there are two classes of reflections: the ones in lines through a pair of opposite vertices, and the ones in lines through the middles of opposite sides. See figure for two conjugacy classes of reflections in D_6:



2. Use problem 1 to describe all 1-dimensional representations of D_n.

Solution 1: Let $\rho : D_n \to GL(1,\mathbb{C}))$ be a 1-dimensional representation, and R_ℓ a reflection in D_n. We have $R_\ell^2 = 1$, so if $a = \rho(R_\ell)$ then $a^2 = 1$ and $a = \pm 1$.

If n is odd, then all reflections are conjugate to each other, and either all of them are sent to $+1$, or all of them are sent to -1. In both cases, all rotations are sent to 1, and we get two different 1-dimensional representations.

If n is even, then there are two conjugacy classes of reflections, and each class is sent to ± 1. If both conjugacy classes are sent to $+1$ or both to -1, all rotations are sent to 1. If one conjugacy class is sent to $+1$ and another to -1, then all rotations by even multiples of $\frac{2\pi}{n}$ are sent to 1, and all rotations by odd multiples of $\frac{2\pi}{n}$ are sent to -1. In total, there are 4 different 1-dimensional representations.

Solution 2: We use the standard presentation D_n by generators and relations: $x^n = 1, y^2 = 1, yx = x^{-1}y$. Here x is a rotation and y is a reflection. Let $\rho(x) = a$ and $\rho(y) = b$, then

$$a^n = 1, b^2 = 1, ab = a^{-1}b,$$

so $a = a^{-1}$ and $a^2 = 1$. We have $a = \pm 1$ and $b = \pm 1$. If n is odd and $a = -1$, then $a^n \neq 1$, so a must be equal to 1 while $b = \pm 1$. If n is even, we can have $a = \pm 1$ and $b = \pm 1$, so there are four possible cases.

3. Recall that the **averaging** operator for a representation $\rho : G \to GL(n)$ is defined as

$$\mathrm{Av}_G = \frac{1}{|G|}\sum_{g\in G}\rho(g).$$

Compute $\mathrm{Av}_{S_3}(v)$ where $v = (x_1, x_2, x_3)$ is a vector in the 3-dimensional permutation representation of S_3.

Solution: We have

$$\mathrm{Av}_{S_3}(v) = \frac{1}{6}\left[(x_1,x_2,x_3)+(x_2,x_1,x_3)+(x_3,x_2,x_1)+(x_1,x_3,x_2)+(x_2,x_3,x_1)+(x_3,x_1,x_2)\right] =$$

$$\frac{1}{6}(2x_1{+}2x_2{+}2x_3, 2x_1{+}2x_2{+}2x_3, 2x_1{+}2x_2{+}2x_3) = \left(\frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3}{3}\right).$$

4. Prove that for any $n > 1$ the sum of all complex roots of unity of degree n equals 0. *Hint: Use a one-dimensional representation of the cyclic group of order n.*

Proof: Consider the cyclic group G_n with generator x of order n and its one-dimensional representation where

$$\rho(x) = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n}) = e^{\frac{2\pi i}{n}}.$$

Then $\rho(x^k) = \rho(x)^k$ runs over all complex roots of unity of degree k.

Let us apply the averaging operator to the vector 1:

$$\mathrm{Av}_{G_n}(1) = \frac{1}{n}(\rho(1)\cdot 1 + \rho(x)\cdot 1 + \cdots + \rho(x^{n-1})\cdot 1) =$$

$$\frac{1}{n}(\rho(1) + \rho(x) + \cdots + \rho(x^{n-1})).$$

Suppose that the sum of all complex roots of unity of degree n is not equal to 0, then by a theorem from lecture $\mathrm{Av}_{G_n}(1)$ spans a 1-dimensional G_n-invariant subspace of $\mathbb{C}$ which is impossible (since the representation is irreducible). Contradiction, therefore the sum of roots of unity equals 0.