MAT 150C, Spring 2021
Solutions to homework 3

1. Compute the character of the 4-dimensional permutation representation of S_4.

Solution: Recall that for any representation corresponding to a group action the character of an element g equals the number of fixed points of g. We have the following numbers of fixed points:

- The identity permutation has 4 fixed points
- A transposition (cycle of length 2) has 2 fixed points
- A cycle of length 3 has 1 fixed point
- A cycle of length 4 has no fixed points
- A product of two disjoint transpositions (like (1 2)(3 4)) has no fixed points.

2. The group D_4 acts on the set of vertices of a square. Compute the character of the corresponding representation.

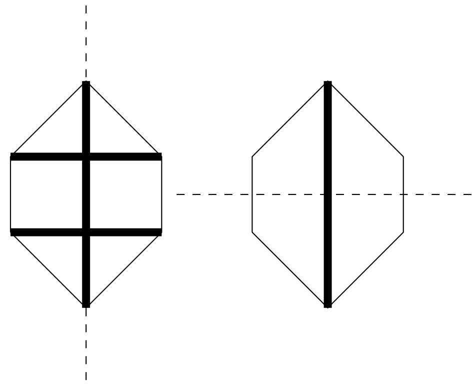
Solution: The group D_4 has the following elements:

- Identity has 4 fixed vertices
- Rotation by $\pm\frac{\pi}{2}$ has no fixed vertices
- Rotation by π has no fixed vertices
- Reflection in a diagonal through opposite vertices (there are 2 such diagonals) has 2 fixed vertices
- Reflection in a diagonal through the middle of opposite edges (there are 2 such diagonals) has no fixed vertices

3. The group D_6 acts on the set of diagonals in a hexagon. Compute the character of the corresponding representation.

Solution: The group D_6 has the following elements:

- Identity has 9 fixed diagonals
- Rotation by $\pm\frac{2\pi}{6}$ has no fixed diagonals
- Rotation by $\pm\frac{4\pi}{6}$ has no fixed diagonals
- Rotation by π has 3 fixed (long) diagonals
- Reflection in a diagonal through opposite vertices (there are 3 such diagonals) has 3 fixed diagonals (see picture below)
- Reflection in a diagonal through the middle of opposite edges (there are 3 such diagonals) has 1 fixed diagonal (see picture below).



4. A representation of S_3 has the following character:

1	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
17	-1	-1	-1	2	2

Decompose it into a direct sum of irreducible representations.

Solution: We have the following character table of S_3, the first three rows are characters of irreducible representations and the last row is the character of our unknown representation V:

	1	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
triv	1	1	1	1	1	1
sgn	1	-1	-1	-1	1	1
$\mathbb{C}^2$	2	0	0	0	-1	-1
V	17	-1	-1	-1	2	2

We have

$\langle \chi_{triv}, \chi_V \rangle = \frac{1}{6}(1\cdot 17+1\cdot(-1)+1\cdot(-1)+1\cdot(-1)+1\cdot 2+1\cdot 2) = \frac{18}{6} = 3,$

$\langle \chi_{sgn}, \chi_V \rangle = \frac{1}{6}(1\cdot 17+(-1)\cdot(-1)+(-1)\cdot(-1)+(-1)\cdot(-1)+1\cdot 2+1\cdot 2) = \frac{24}{6} = 4,$

$\langle \chi_{\mathbb{C}^2}, \chi_V \rangle = \frac{1}{6}(2\cdot 17+0\cdot(-1)+0\cdot(-1)+0\cdot(-1)+(-1)\cdot 2+(-1)\cdot 2) = \frac{30}{6} = 5,$

Therefore

$$V = 3(\text{triv}) \oplus 4(\text{sgn}) \oplus 5(\mathbb{C}^2).$$