MAT 150C, Spring 2021
Solutions to homework 4

1. Consider the permutation representation $S_3 \to GL(V),\ V = \mathbb{C}^3$.
(a) Compute the dimension of the space of S_3-invariant transformations from V to V.
(b) Find an explicit basis in this space.

Solution: (a) The character of the permutation representation is given by the following table:

1	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
3	1	1	1	0	0

The dimension of the space of S_3-invariant linear transformations equals:

$$(\chi, \chi) = \frac{1}{6}(3 \cdot 3 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = 2.$$

(b) The permutation representation splits into irreducibles as $\mathbb{C}^3 = U \oplus U^\perp$ where U is the copy of the trivial representation spanned by the vector $(1, 1, 1)$. By Schur's lemma, the space of S_3-invariant linear transformations is spanned by the projections to U and $U^\perp$. The projection to U is given by averaging

$$(x_1, x_2, x_3) \mapsto \left(\frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + x_3}{3}\right)$$

and the corresponding matrix is

$$T_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

The projection to $U^\perp$ is given by the matrix

$$T_2 = I - T_1 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

To sum up, the space of S_3-invariant linear transformations is spanned by T_1 and T_2.

2. Prove that complex numbers with absolute value 1 have the form $\{e^{i\theta}, \theta \in \mathbb{R}\}$ and form a group.

Solution: A complex number with absolute value r and argument θ can be written as

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.$$

For $r = 1$ we get $z = e^{i\theta}$. Let us prove that these form a group. Indeed,

$$1 = e^{i \cdot 0},\ e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)},\ (e^{i\theta})^{-1} = e^{i(-\theta)}.$$

This set contains 1 and is closed under multiplication and taking inverses, so it is a group.

3. Find all **continuous** one-dimensional representations of the group from problem 2.

Solution: Since $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, it is easy to see that $\rho : e^{i\theta} \mapsto e^{ik\theta}$ is a group homomorphism. Since $\rho(e^{2\pi i}) = \rho(1) = 1$, we get $e^{2\pi ik} = 1$, so k must be an integer.

4. The character of a representation of $SU(2)$ has the form

$$\chi \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \alpha^{-2} + 3\alpha^{-1} + 2 + 3\alpha + \alpha^2.$$

Decompose this representation into irreducibles.

Solution: Recall that the characters of the irreducible representations V_k have the form:

$$\chi(V_0) = 1,\ \chi(V_1) = \alpha + \alpha^{-1},\ \chi(V_2) = \alpha^2 + 1 + \alpha^{-2}.$$

Therefore

$$\chi = (\alpha^{-2}+3\alpha^{-1}+2+3\alpha+\alpha^2) = (\alpha^2+1+\alpha^{-2})+3(\alpha+\alpha^{-1})+1 = \chi(V_2)+3\chi(V_1)+\chi(V_0),$$

so our representation is isomorphic to $V_2 \oplus 3V_1 \oplus V_0$.