MAT 150C, Spring 2021
Solutions to homework 7

1. Suppose that $\cos(\alpha)$ is a constructible number. Prove that $\sin(\alpha)$ is constructible.

Solution: We have $\sin(\alpha) = \sqrt{1-\cos^2(\alpha)}$. Since $\cos(\alpha)$ is constructible, and the product of constructible numbers is constructible, we have that $\cos^2(\alpha)$ and $1-\cos^2(\alpha)$ are constructible. Since the square root of a constructible number is constructible, $\sin(\alpha)$ is also constructible.

2. Suppose that $\cos(\alpha)$ is a constructible number. Prove that $\cos(2\alpha)$ and $\cos(\frac{\alpha}{2})$ are constructible.

Solution: We have $\cos(2\alpha) = 2\cos^2(\alpha) - 1$ and $\cos(\frac{\alpha}{2}) = \sqrt{\frac{1+\cos(\alpha)}{2}}$, so similarly to problem 1 these are constructible.

3. Let $z = e^{\frac{2\pi i}{5}}$ be the fifth root of unity, and $x = \cos(\frac{2\pi}{5})$.

a) Prove that $z + z^4 = 2x$, and $z^2 + z^3 = 2(2x^2 - 1)$.

b) Use the equation $1 + z + z^2 + z^3 + z^4 = 0$ and part (a) to find an algebraic equation for x. Solve it and find an explicit formula for x.

Solution: a) We have $z = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})$ and

$$z^4 = \cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5}) = \cos(\frac{2\pi}{5}) - i\sin(\frac{2\pi}{5}),$$

so $z + z^4 = 2\cos(\frac{2\pi}{5}) = 2x$. Similarly, $z^2 + z^3 = 2\cos(\frac{4\pi}{5})$ and by double angle formula we have

$$\cos(\frac{4\pi}{5}) = 2\cos^2(\frac{2\pi}{5}) - 1 = 2x^2 - 1.$$

b) We have

$$0 = 1+z+z^2+z^3+z^4 = 1+(z+z^4)+(z^2+z^3) = 1+2x+2(2x^2-1) = 4x^2+2x-1.$$

Therefore $x = \frac{-2\pm\sqrt{20}}{8} = \frac{-1\pm\sqrt{5}}{4}$. Since $x > 0$, we get $x = \frac{-1+\sqrt{5}}{4}$.

4. Use problem 3 to construct a regular pentagon by ruler and compass.

Solution: By Problem 3 we have $x = \cos(\frac{2\pi}{5}) = \frac{-1+\sqrt{5}}{4}$ is a constructible number, and we can construct a point $A = (\cos(\frac{2\pi}{5}), (\sin(\frac{2\pi}{5}))$. Together with $B = (1, 0)$, these two points are vertices of a regular pentagon, and we can use compass to draw circles with radius AB and construct all other vertices of the pentagon.