MAT 17A, solutions to practice problems for the final exam

1. Compute the limit:

   a) \( \lim_{x \to 3} e^{1/x} \)

   Answer: \( e^{1/3} \).

   b) \( \lim_{x \to 3} \ln(x - 3) \)

   Answer: \(-\infty\).

   c) \( \lim_{x \to 3} \frac{\ln(x-2)}{x-3} \)

   Solution: As \( x \) approaches 3, \((x-2)\) approaches 1, so \(\ln(x-2)\) approaches \(\ln(1) = 0\). Therefore we have a limit of the form 0/0 and can apply the L'Hôpital's rule:

   \[
   \lim_{x \to 3} \frac{\ln(x-2)}{x-3} = \lim_{x \to 3} \frac{1/(x-2)}{1} = 1.
   \]

   d) \( \lim_{x \to \infty} \frac{8x^5 - 7x^3 + 9}{(3x^2-1)(2x^3-3)} \)

   Solution: Let us divide the top and the bottom of this fraction by \(x^5\):

   \[
   \lim_{x \to \infty} \frac{8x^5 - 7x^3 + 9}{(3x^2-1)(2x^3-3)} = \lim_{x \to \infty} \frac{8 - 7/x^2 + 9/x^5}{(3 - 3/x^2)(2 - 3/x^3)} = \frac{8 - 0 + 0}{(3 - 0)(2 - 0)} = \frac{4}{3}.
   \]

   e) \( \lim_{x \to \infty} \frac{e^x}{x^3} \)

   Solution: As this is the limit of the type \(\infty/\infty\), we can apply the L'Hôpital’s rule several times:

   \[
   \lim_{x \to \infty} \frac{e^x}{x^3} = \lim_{x \to \infty} \frac{e^x}{3x^2} = \lim_{x \to \infty} \frac{e^x}{6x} = \lim_{x \to \infty} \frac{e^x}{6} = \infty.
   \]

   f) \( \lim_{x \to 0} \frac{\arctan(x) - x}{x^4} \)

   Solution: As this is the limit of the type 0/0, we can apply the L'Hôpital’s rule several times:

   \[
   \lim_{x \to 0} \frac{\arctan(x) - x}{x^3} = \lim_{x \to 0} \frac{1/(1 + x^2) - 1}{3x^2} = \lim_{x \to 0} \frac{1 - 1 - x^2}{3x^2(1 + x^2)} = \lim_{x \to 0} \frac{-x^2}{3(1 + x^2)} = \lim_{x \to 0} \frac{-1}{3} = -\frac{1}{3}.
   \]
2. Compute the derivative of the following functions:
   a) \( f(x) = x \ln x - x \)
   \textbf{Answer:} \( \ln(x) \)
   b) \( f(x) = e^{3x^2} \)
   \textbf{Answer:} \( 6xe^{3x^2} \).
   c) \( f(x) = (x - 1)^5 \cos x \)
   \textbf{Answer:} \( 5(x - 1)^4 \cos x - (x - 1)^5 \sin x \).
   d) \( f(x) = \sin(\ln x) \)
   \textbf{Answer:} \( \cos(\ln x) \cdot x \).
   e) \( f(x) = e^{3x^2} + 2 \cos x + 2 \)
   \textbf{Answer:} \( (3 \cos x + 6 + \sin x)e^{3x^2} \).

3. Find the minimal and maximal values of a function:
   a) \( f(x) = x^2 e^{-x} \) on \([0, 1]\)
   \textbf{Solution:} We have \( f'(x) = 2xe^{-x} - x^2e^{-x} = x(2 - x)e^{-x} \), so \( f'(x) = 0 \) for \( x = 0 \) and \( x = 2 \). Since \( x = 2 \) is not on the interval and \( x = 0 \) is one of the endpoints, we just need to compute \( f(0) = 0 \) with \( f(1) = e^{-1} \). Therefore the minimal value is 0 and the maximal value is \( e^{-1} \).
   b) \( f(x) = 2x^3 - 3x^2 + 1 \) on \([-1, 2]\)
   \textbf{Solution:} We have \( f'(x) = 6x^2 - 6x = 6x(x - 1) \), so \( f'(x) = 0 \) for \( x = 0 \) and \( x = 1 \). We need to compute
   \[
   f(-1) = -2 - 3 + 1 = -4, \quad f(0) = 1, \quad f(1) = 2 - 3 + 1 = 0, \quad f(2) = 16 - 12 + 1 = 5.
   \]
   Therefore the minimal value is \((-4)\) and the maximal value is 5.
   c) \( f(x) = \sin^2 x \) on \([0, \pi]\)
   \textbf{Solution:} We have \( f'(x) = 2 \sin x \cos x \), so \( f'(x) = 0 \) when either \( \sin x = 0 \) (so \( x = \pi k \)) or \( \cos x = 0 \) (so \( x = \pi / 2 + \pi k \)). On the interval \([0, \pi]\) we have 3 critical numbers \( 0, \pi / 2, \pi \) and both endpoints are among them. Therefore one needs to compute \( f(0) = 0, \ f(\pi / 2) = 1, \ f(\pi) = 0 \), and the minimal and maximal values are 0 and 1 respectively.
   d) \( \frac{x}{1+x^2} \) on \([-2, 2]\)
Solution: We have \( f'(x) = \frac{1+x^2-x(2x)}{(1+x^2)^2} = 1 - x^2(1 + x^2)^2 \), and the critical numbers are \( x = 1 \) and \( x = -1 \). We have

\[
\begin{align*}
    f(-2) &= -2/5, \\
    f(-1) &= -1/2, \\
    f(1) &= 1/2, \\
    f(2) &= 2/5.
\end{align*}
\]

Since \( 1/2 > 2/5 \), the minimal and maximal values are \(-1/2\) and \(1/2\) respectively.

4. Find the equation of the tangent line to the graph of \( f(x) = \ln x \) at \( x = 5 \).

Answer: \( y = \frac{1}{5}x + (\ln 5 - 1) \).

5. For a given function:

- Find the domain
- Determine the equations of vertical and horizontal asymptotes
- Find the derivative and determine the intervals where the function is increasing/decreasing
- Find the second derivative and determine the intervals where the function is concave up/down, find inflection points
- Draw the graph using all the information above

a) \( f(x) = 2x^3 - 3x^2 + 1 \)

Solution: The function is defined everywhere, and there are no asymptotes. We have \( f'(x) = 6x^2 - 6x = 6x(x-1) \), so the function is increasing for \( x > 1 \) and \( x < 0 \), and decreasing for \( 0 < x < 1 \). Furthermore, \( f''(x) = 12x-6 \), so the function is concave up for \( x > 1/2 \) and concave down for \( x < 1/2 \), and it has an inflection point at \( x = 1/2 \).
b) $f(x) = xe^{-x}$

**Solution:** The function is defined everywhere. To find the asymptotes, remark that at $x \to -\infty$ the function $e^{-x}$ goes to infinity, so $\lim_{x \to -\infty} xe^{-x} = -\infty$. To find the limit at $x \to +\infty$, let us use the L'Hôpital’s rule:

$$
\lim_{x \to +\infty} xe^{-x} = \lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0.
$$

Therefore the graph has a horizontal asymptote $y = 0$ at $x \to +\infty$.

Now $f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}$, so the function increases for $x < 1$ and decreases for $x > 1$. Furthermore, $f''(x) = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$, so the function has an inflection point at $x = 2$.

c) $f(x) = \ln(x^2 + 1)$
**Solution:** Since $x^2 + 1 \geq 1$, the function is defined and nonnegative everywhere. As $x$ approaches $\pm\infty$, $x^2 + 1 \to +\infty$, so $\ln(x^2 + 1) \to +\infty$, and there are no horizontal asymptotes.

Now $f'(x) = \frac{2x}{x^2+1}$, so the function decreases for $x < 0$ and increases for $x > 0$. Furthermore,

$$f''(x) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{2(1 - x^2)}{(x^2 + 1)^2},$$

and the function has inflection points at $x = \pm 1$.

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d) $f(x) = \frac{x-1}{x+1}$

**Solution:** The function is defined for $x \neq -1$, and has a vertical asymptote at $x = -1$. To find horizontal asymptotes, we have

$$\lim_{x \to \infty} \frac{x - 1}{x + 1} = \lim_{x \to \infty} \frac{1 - 1/x}{1 + 1/x} = 1.$$

Now $f'(x) = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$, so the function increases on every interval where it is defined. Furthermore, $f''(x) = -4(x + 1)^3$, so the function is concave down for $x > -1$ and concave up for $x < -1$. 

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6. Consider the function

\[ f(x) = \begin{cases} 
  x + 1, & \text{if } x < -1 \\
  x^2 + ax + b, & \text{if } x \geq -1.
\end{cases} \]

a) For which values of the parameters it is continuous?

**Solution:** The function is continuous, if its limits from the left and from the right at \( x = -1 \) coincide:

\[ (-1) + 1 = (-1)^2 + a(-1) + b \iff 0 = 1 - a + b \iff b = a - 1. \]

b) For which values of the parameters it has a derivative at every point?

**Solution:** The function clearly has a derivative for all \( x \neq -1 \), and it has derivative at \( x = -1 \) if it is continuous at this point (see (a)), and the derivatives from the left and from the right coincide: \( 1 = 2(-1) + a \), so \( a = 3 \) and \( b = 2 \).  

7. Consider the curve given by the equation \( x^{2/3} + y^{2/3} = 1 \). Find \( y' \) using implicit differentiation and sketch this curve. (Note: assume that \( x^{2/3} = \sqrt[3]{x^2} \) is defined for all \( x \)).

**Answer:** \( y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}. \)

8. Consider the curve given by the equation \( y^2 = x^3 - x \).

a) Find \( y' \) using implicit differentiation.
b) Find the equation of the tangent line at the point \((2, \sqrt{6})\).

**Answer:**

a) \(y' = \frac{3x^2 - 1}{2y}\).

b) \(y = \frac{11}{2\sqrt{6}}(x - 2) + \sqrt{6}\).

9. An open rectangular box with square base is to be made from 1 area unit of material. What dimensions will result in a box with the largest possible volume?

**Solution:** Let \(h\) be the height of the box and let \(x\) be the size of the base. Then the surface area equals \(x^2 + 4xh = 1\), so

\[
h = \frac{(1 - x^2)}{4x}.
\]

The volume then equals \(V(x) = x^2h = x(1 - x^2)/4 = (x - x^3)/4\). We get \(V'(x) = (1 - 3x^2)/4\), so the maximal volume is at

\[
x = \frac{1}{\sqrt{3}}, \quad h = \frac{1 - 1/3}{4/\sqrt{3}} = \frac{2\sqrt{3}}{12} = \frac{1}{2\sqrt{3}}.
\]

10. A TV set costs $100. If its price is lowered by \(a\)% the sales would increase by \(2a\)%? Find the discount amount \(a\) which yields the maximal profit.

**Answer:** 25%