Math 17A, Fall 2023
Solutions to practice problems for Midterm 1

1. Consider the functions:

$$f(x) = \frac{x^2+1}{x^2-1},\ g(x) = ||x+1|-1|,\ h(x) = \frac{1}{\ln x}.$$

For each of them:

a) Find the domain

b) Determine the intersection points of the graph with the coordinate axis

c) Find all vertical and horizontal asymptotes (if any)

d) Sketch the graph

Solution: 1) The function $f(x)$ is defined for $x^2 - 1 \neq 0$, so that $x \neq \pm 1$. The domain is

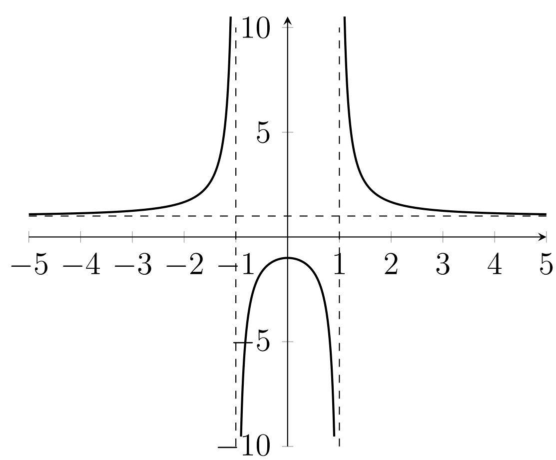
$$(-\infty, -1) \cup (-1, 1) \cup (1, +\infty).$$

The intesection with the y-axis is at $f(0) = -1$. The intersection with the x-axis is where $f(x) = 0$ which is not possible since $x^2 + 1 > 0$. Note that $f(x) > 0$ for $x > 1$ and $x < -1$, and $f(x) < 0$ on $(-1, 1)$.

At $x = \pm 1$ we have $x^2 + 1 \to 2$ and $x^2 - 1 \to 0$, so there are vertical asymptotes at $x = \pm 1$. For the horizontal asymptote, we have

$$\lim_{x\to\pm\infty} \frac{x^2+1}{x^2-1} = \lim_{x\to\pm\infty} \frac{1+1/x^2}{1-1/x^2} = 1.$$

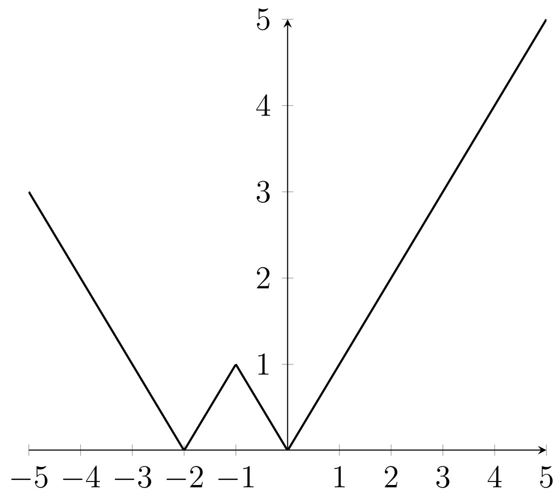
Therefore we have a horizontal asymptote $y = 1$.



2) The function $g(x)$ is defined everywhere, so the domain is $(-\infty, +\infty)$. There are no vertical asymptotes. We can write

$$g(x) = \begin{cases} x, x \geq 0 \\ -x, -1 \leq x \leq 0 \\ 2 + x, -2 \leq x \leq -1 \\ -2 - x, x \leq -2. \end{cases}$$

So there are no horizontal asymptotes either.



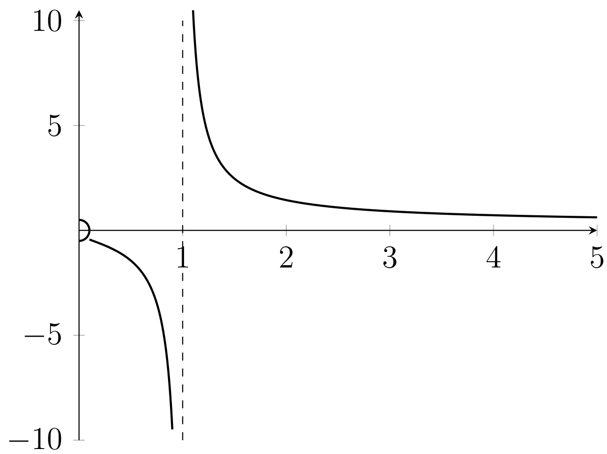
3) The function $h(x)$ is defined where $\ln x$ is defined (so $x > 0$) and $\ln x \neq 0$ (so $x \neq 1$). Therefore the domain is $(0, 1) \cup (1, +\infty)$. To find the vertical asymptotes, let us compute the limits at 0 and at 1:

$$\lim_{x \to 0} \ln x = -\infty, \ \lim_{x \to 0} \frac{1}{\ln x} = 0;$$

$$\lim_{x \to 1} \ln x = 0, \ \lim_{x \to 0} \frac{1}{\ln x} = \infty.$$

To find the horizontal asymptotes, we get

$$\lim_{x \to +\infty} \ln x = +\infty, \ \lim_{x \to +\infty} \frac{1}{\ln x} = 0.$$



2. For which c the function

$$f(x) = \begin{cases} \sin x, & \text{if } x < c, \\ \cos x, & \text{if } x \geq c \end{cases}$$

is continuous?

Solution: The function is clearly continuous for $x \neq c$. At $x = c$, we have the following one-sided limits:

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} \cos x = \cos(c), \quad \lim_{x \to c^-} f(x) = \lim_{x \to c^-} \sin x = \sin(c)$$

The function has a limit at c (and is continuous at c) if these one-sided limits agree:

$$\cos(c) = \sin(c), \quad c = \frac{\pi}{4} + \pi k, \quad k \text{ integer.}$$

3. Compute the limits:

a)

$$\lim_{x \to 0} (\sin^3 x - \cos x)$$

Solution:

$$\lim_{x \to 0} (\sin^3 x - \cos x) = \sin^3(0) - \cos(0) = 0^3 - 1 = -1.$$

b)

$$\lim_{x \to 0} \frac{\sqrt{x+1}}{\sqrt{x}}$$

Solution: We have $\lim_{x\to 0}\sqrt{x+1} = \sqrt{0+1} = 1$, $\lim_{x\to 0}\sqrt{x} = 0$, so

$$\lim_{x\to 0}\frac{\sqrt{x+1}}{\sqrt{x}} = \infty.$$

c)

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-3x}.$$

Solution: We have

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-3x} = \lim_{x\to 3}\frac{(x-3)(x+1)}{x(x-3)} = \lim_{x\to 3}\frac{x+1)}{x} = \frac{4}{3}.$$

d)

$$\lim_{x\to 0}\frac{7x^3+8x^2+9x}{11x+12x^2-13x^3}.$$

Solution: We have

$$\lim_{x\to 0}\frac{7x^3+8x^2+9x}{11x+12x^2-13x^3} = \lim_{x\to 0}\frac{x(7x^2+8x+9)}{x(11+12x-13x^2)} =$$

$$\lim_{x\to 0}\frac{(7x^2+8x+9)}{(11+12x-13x^2)} = \frac{9}{11}.$$

e)

$$\lim_{x\to +\infty} e^{-x^2}$$

Solution: If $x \to +\infty$ then $x^2 \to +\infty$, $-x^2 \to -\infty$ and hence $\lim_{x\to+\infty} e^{-x^2} = 0$ (since $\lim_{x\to -\infty} e^x = 0$).

f)

$$\lim_{x\to +\infty}(\ln(x+1) - \ln x)$$

Solution: We have

$$\lim_{x\to +\infty}(\ln(x+1) - \ln x) = \lim_{x\to +\infty}\ln\left(\frac{x+1}{x}\right).$$

Now $\lim_{x\to+\infty}\frac{x+1}{x} = \lim_{x\to+\infty}(1+\frac{1}{x}) = 1$, so

$$\lim_{x\to +\infty}\ln\left(\frac{x+1}{x}\right) = \ln(1) = 0.$$

g)

$$\lim_{x\to+\infty}\frac{7x^2+8x+9}{11+12x-13x^2}.$$

Solution: Let us divide the numerator and the denominator of this fraction by x^2, which is the highest power of x here:

$$\lim_{x\to+\infty}\frac{7x^2+8x+9}{11+12x-13x^2}=\lim_{x\to+\infty}\frac{7+8/x+9/x^2}{11/x^2+12/x-13}=\frac{7+0+0}{0+0-13}=-\frac{7}{13}.$$

h)

$$\lim_{x\to+\infty}\frac{2x^3+\sqrt{x^4+7}}{3x^3-5}$$

Solution: Let us divide the numerator and denominator by x^3:

$$\frac{2x^3+\sqrt{x^4+7}}{3x^3-5}=\frac{2+\frac{1}{x^3}\sqrt{x^4+7}}{3-5/x^3}=$$

$$\frac{2+\frac{1}{\sqrt{x^6}}\sqrt{x^4+7}}{3-5/x^3}=\frac{2+\sqrt{1/x^2+7/x^6}}{3-5/x^3}.$$

Therefore

$$\lim_{x\to+\infty}\frac{2x^3+\sqrt{x^4+7}}{3x^3-5}=\frac{2+\sqrt{0+0}}{3-0}=\frac{2}{3}.$$

4. Find $\lim_{x\to1} f(x)$, where

$$f(x)=\begin{cases}x^2+3x+4, & \text{if } x\ge 1,\\ 7x^2+1, & \text{if } x<1.\end{cases}$$

Solution: We find one-sided limits:

$$\lim_{x\to1^+} f(x)=\lim_{x\to1^+}(x^2+3x+4)=8,$$

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}(7x^2+1)=8,$$

so $\lim_{x\to1} f(x)=8$.

5. Plot the graph of the function

$$\operatorname{sgn}(x)=\begin{cases}\frac{|x|}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0.\end{cases}$$

Is it continuous?

Solution: The function equals 1 for $x > 0$, -1 for $x < 0$ and 0 at $x = 0$. It is not continuous at $x = 0$.

6. For the function

$$f(x) = \begin{cases} x^2 + x - 9, & \text{if } |x| \geq 3, \\ x, \text{ if } |x| < 3, \end{cases}$$

find one-sided limits at $x = -3$, $x = 0$, $x = 3$. Is this function continuous?

Solution: We find one-sided limits:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2 + x - 9) = 9 + 3 - 9 = 3,$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x = 3,$$

$$\lim_{x \to (-3)^+} f(x) = \lim_{x \to (-3)^+} x = -3,$$

$$\lim_{x \to (-3)^-} f(x) = \lim_{x \to (-3)^-} (x^2 + x - 9) = 9 - 3 - 9 = -3,$$

so the function is continuous both at $x = 3$ and at $x = -3$. Since it is clearly continuous for $x \neq \pm 3$, it is continuous everywhere. The limit at $x = 0$ equals 0.

7*. Use the Intermediate Value Theorem to show that the equation

$$x^3 + x - 3 = 0$$

has a solution on the interval $[1, 2]$.

Solution: Let $f(x) = x^3 + x - 3$, then $f(1) = 1 + 1 - 3 = -1 < 0$, $f(2) = 8 + 2 - 3 = 7 > 0$. By Intermedate Value theorem, there is a point x on $[1, 2]$ such that $f(x) = 0$.