In all problems:

- Find the domain, vertical and horizontal asymptotes
- Find the intervals where the function is increasing or decreasing
- Find all local maximums and minimums
- Find the intervals where the function is concave up or down, and the inflection points
- Graph the function using this information

1. \( f(x) = 5 - 2x - x^2 \)

   **Solution:** The function is defined and continuous for all real numbers, so there are no vertical asymptotes. At \( x \to \pm \infty \) we get \( f(x) \to -\infty \), so there are no horizontal asymptotes either.

   Next, \( f'(x) = -2 - 2x = -2(x + 1) \). If \( x \geq -1 \) then \( x + 1 \geq 0 \) and \( f'(x) < 0 \), so \( f(x) \) is decreasing on \([-1, +\infty)\). If \( x \leq -1 \), then \( x + 1 \leq 0 \) and \( f'(x) > 0 \), so \( f(x) \) is increasing on \((-\infty, -1]\). The function has a local maximum at \( x = -1 \).

   The second derivative equals \( f''(x) = -2 \), so \( f(x) \) is concave down everywhere. There are no inflection points.

![Graph of f(x) = 5 - 2x - x^2](image)

2. \( f(x) = x^4 + 2x^3 \)

   **Solution:** The function is defined and continuous for all real numbers, so there are no vertical asymptotes. At \( x \to \pm \infty \) we get \( f(x) \to \infty \), so there are no horizontal asymptotes either.

   Next, \( f'(x) = 4x^3 + 6x^2 = x^2(4x + 6) \). Note that \( x^2 \geq 0 \) for all \( x \), and \( 4x + 6 \geq 0 \) for \( x \geq -\frac{3}{2} \). Therefore \( f(x) \) is increasing on \([-\frac{3}{2}, +\infty)\) and decreasing on \((-\infty, -\frac{3}{2}]\). The function has a local maximum at \( x = -\frac{3}{2} \).

   The second derivative equals \( f''(x) = 12x^2 + 12x = 12x(x + 1) \). We have \( f''(x) \geq 0 \) for \( x \geq 0 \) and \( x \leq -1 \), and \( f''(x) \leq 0 \) for \(-1 \leq x \leq 0 \). Therefore the function is concave down on \([-1, 1]\) and concave up on \((-\infty, -1] \cup [0, +\infty)\). There are inflection points at \( x = -1 \) and \( x = 0 \).
3. \( f(x) = \frac{e^x}{x} \)

**Solution:** The function is defined for \( x \neq 0 \). Since \( \lim_{x \to 0} \frac{e^x}{x} = \infty \), there is a vertical asymptote at \( x = 0 \). To find horizontal asymptotes, we compute the limits at \( \pm \infty \):

1) \( \lim_{x \to \infty} \frac{e^x}{x} = (\text{L'Hospital Rule}) = \lim_{x \to \infty} \frac{e^x}{1} = +\infty. \)
2) \( \lim_{x \to -\infty} e^x = 0, \) so \( \lim_{x \to -\infty} \frac{e^x}{x} = 0. \)

Therefore the function has a horizontal asymptote \( y = 0 \) at \( x \to -\infty \).

The first derivative equals (by Quotient Rule):

\[
 f'(x) = \frac{e^x \cdot x - e^x \cdot 1}{x^2} = \frac{e^x (x-1)}{x^2}.
\]

Since \( e^x > 0 \) and \( x^2 > 0 \) for all \( x \neq 0 \), we have \( f'(x) > 0 \) for \( x > 1 \) and \( f'(x) < 0 \) for \( x < 1 \). Therefore the function is decreasing on \((-\infty, 0) \cup (0, 1]\) and increasing on \([1, +\infty)\).

There is a local minimum at \( x = -1 \).

The second derivative equals

\[
 f''(x) = \frac{e^x (x-1) + e^x \cdot 1 \cdot x^2 - e^x (x-1) \cdot 2x}{x^4} = \frac{e^x (x^3 - 2x^2 + 2x)}{x^4} = \frac{e^x (x^3 - 2x^2 + 2x)}{x^4}
\]

Here we used that \( [e^x (x-1)]' = e^x (x-1) + e^x \cdot 1 \) by Product Rule. Now \( x^3 - 2x^2 + 2x = x(x^2 - 2x + 2) \) and \( x^2 - 2x + 2 = (x-1)^2 + 1 > 0 \), also \( e^x > 0 \) and \( x^4 > 0 \) for all \( x \neq 0 \). Therefore \( f''(x) > 0 \) for \( x > 0 \) and \( f''(x) < 0 \) for \( x < 0 \), so \( f(x) \) is concave down on \((-\infty, 0)\) and concave up on \((0, +\infty)\).