MAT 17A Fall 2023
Solutions to homework 7

In all problems:

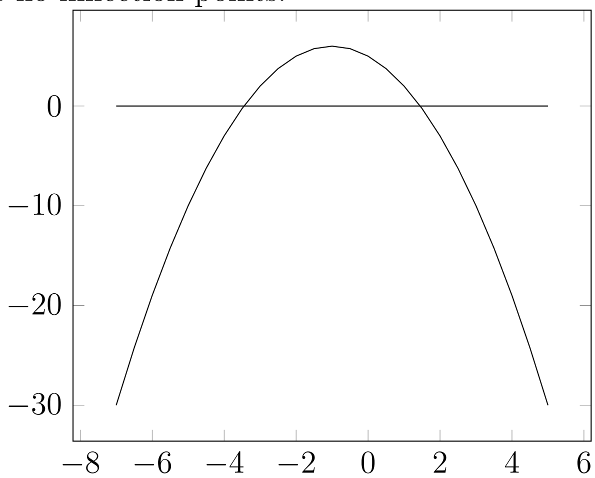
- Find the domain, vertical and horizontal asymptotes
- Find the intervals where the function is increasing or decreasing
- Find all local maximums and minimums
- Find the intervals where the function is concave up or down, and the inflection points
- Graph the function using this information

1. $f(x) = 5 - 2x - x^2$

Solution: The function is defined and continuous for all real numbers, so there are no vertical asymptotes. At $x \to \pm\infty$ we get $f(x) \to -\infty$, so there are no horizontal asymptotes either.

Next, $f'(x) = -2 - 2x = -2(x+1)$. If $x \geq -1$ then $x + 1 \geq 0$ and $f'(x) < 0$, so $f(x)$ is decreasing on $[-1, +\infty)$. If $x \leq -1$, then $x + 1 \leq 0$ and $f'(x) > 0$, so $f(x)$ is increasing on $(-\infty, -1]$. The function has a local maximum at $x = -1$.

The second derivative equals $f''(x) = -2$, so $f(x)$ is concave down everywhere. There are no inflection points.

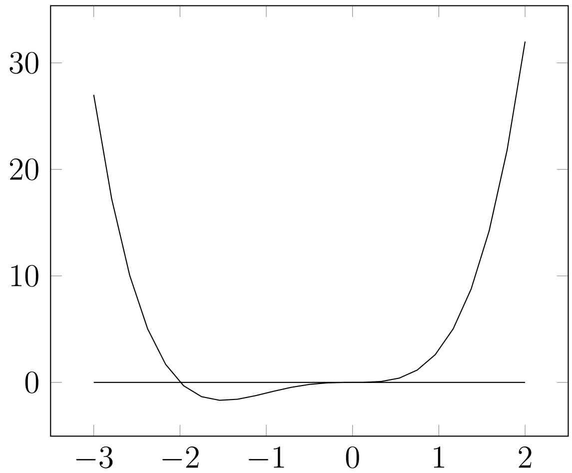


2. $f(x) = x^4 + 2x^3$

Solution: The function is defined and continuous for all real numbers, so there are no vertical asymptotes. At $x \to \pm\infty$ we get $f(x) \to \infty$, so there are no horizontal asymptotes either.

Next, $f'(x) = 4x^3 + 6x^2 = x^2(4x + 6)$. Note that $x^2 \geq 0$ for all x, and $4x + 6 \geq 0$ for $x \geq -\frac{3}{2}$. Therefore $f(x)$ is increasing on $[-\frac{3}{2}, +\infty)$ and decreasing on $(-\infty, -\frac{3}{2}]$. The function has a local minimum at $x = -\frac{3}{2}$.

The second derivative equals $f''(x) = 12x^2 + 12x = 12x(x + 1)$. We have $f''(x) \geq 0$ for $x \geq 0$ and $x \leq -1$, and $f''(x) \leq 0$ for $-1 \leq x \leq 0$. Therefore the function is concave down on $[-1, 1]$ and concave up on $(-\infty, -1] \cup [0, +\infty)$. There are inflection points at $x = -1$ and $x = 0$.



3. $f(x) = \frac{e^x}{x}$

Solution: The function is defined for $x \neq 0$. Since $\lim_{x\to 0} \frac{e^x}{x} = \infty$, there is a vertical asymptote at $x = 0$. To find horizontal asymptotes, we compute the limits at $\pm\infty$:

1) $\lim_{x\to+\infty} \frac{e^x}{x} =$ (L'Hospital Rule) $= \lim_{x\to+\infty} \frac{e^x}{1} = +\infty$.

2) $\lim_{x\to-\infty} e^x = 0$, so $\lim_{x\to-\infty} \frac{e^x}{x} = 0$.

Therefore the function has a horizontal asymptote $y = 0$ at $x \to -\infty$.

The first derivative equals (by Quotient Rule):

$$f'(x) = \frac{e^x \cdot x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}.$$

Since $e^x > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) > 0$ for $x > 1$ and $f'(x) < 0$ for $x < 1$. Therefore the function is decreading on $(-\infty, 0) \cup (0, 1]$ and increasing on $[1, +\infty)$. There is a local minimum at $x = 1$.

The second derivative equals

$$f''(x) = \frac{[e^x(x-1) + e^x \cdot 1] \cdot x^2 - e^x(x-1) \cdot 2x}{x^4} =$$

$$\frac{e^x(x^3 - x^2 + x^2 - 2x^2 + 2x)}{x^4} = \frac{e^x(x^3 - 2x^2 + 2x)}{x^4} =$$

Here we used that $[e^x(x-1)]' = e^x(x-1) + e^x \cdot 1$ by Product Rule. Now $x^3 - 2x^2 + 2x = x(x^2 - 2x + 2)$ and $x^2 - 2x + 2 = (x-1)^2 + 1 > 0$, also $e^x > 0$ and $x^4 > 0$ for all $x \neq 0$. Therefore $f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$, so $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, +\infty)$.