MAT 17A Fall 2023
Solutions to homework 8

1. Use L'Hospital Rule to compute the limit

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2}.$$

Solution: By L'Hospital Rule we get

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2x}.$$

At $x = 0$ the numerator equals $e^0 - e^0 = 1 - 1 = 0$, and the denominator equals 0, so we can apply L'Hospital Rule again:

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{2} = \frac{1+1}{2} = 1.$$

2. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u $(u < v)$, then the time required to swim a distance L is $L/(v-u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v-u}.$$

where a is the proportionality constant. Determine the value of v that minimizes $E(v)$.

Solution: We compute the derivative $E'(v)$ by Quotient Rule keeping in mind that a, L and u are constant:

$$E'(v) = aL\left(\frac{v^3}{v-u}\right)' = aL\frac{3v^2(v-u) - v^3 \cdot 1}{(v-u)^2} =$$

$$aL\frac{v^2(3v - 3u - v)}{(v-u)^2} = aL\frac{v^2(2v-3u)}{(v-u)^2}.$$

The critical points satisfy $v = 0$ (which makes no sense, fish is not moving) and $2v = 3u$, so $v = \frac{3}{2}u$.

3. A box with a square base and open top must have a volume of $32,000\text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.

Solution: Let x be the side of the base and h is the height of the box. Then $x^2h = 32000$, and $h = \frac{32000}{x^2}$. Now the area without the top equals

$$S(x) = x^2 + 4xh = x^2 + 4x \cdot \frac{32000}{x^2} = x^2 + \frac{128000}{x}.$$

Now

$$S'(x) = 2x - \frac{128000}{x^2} = 0$$

is equivalent to $2x^3 = 128000,\ x^3 = 64000$, and $x = 40\text{cm}$. Now

$$h = \frac{32000}{x^2} = \frac{32000}{1600} = 20\text{cm}.$$