Lecture 1 (1/9)

Monday, January 9, 2023 9:30 AM

https://math.ucdavis.edu/~egorskiy/MAT215A-w23/

Def X, Y = top. spaces

X is homeomorphic to Y if $\exists$ maps

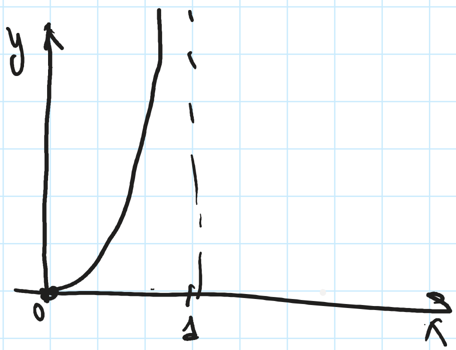
$f : X \to Y$, $g : Y \to X$ both continuous, $f \circ g = Id_Y$

$g \circ f = Id_X$

Ex 1 $X = [0, 1)$ $Y = [0, +\infty)$

$f(x) = \tan\left(\frac{\pi}{2} x\right)$

$g(y) = \frac{2}{\pi} \arctan(y)$



Open ball/disk $\mathring{D}^n = \{ \vec{x} \in \mathbb{R}^n : \|x\| < 1 \}$

$\sum x_i^2 < 1$

$D^n = \{ \vec{x} \in \mathbb{R}^n : \|x\| \le 1 \}$ ← closed ball/disk

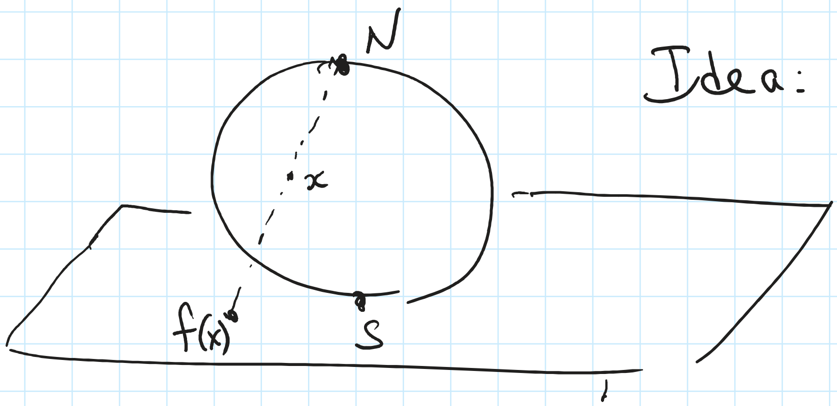
$\partial D^n = S^{n-1}$, $D^n - \partial D^n = \mathring{D}^n$

Ex 2: $\mathring{D}^n \underset{\text{homeo}}{\simeq} \mathbb{R}^n$: apply ex. 1 in radial direction

$$\vec{x} \longrightarrow \frac{\tan\left(\frac{\pi}{2} \|x\|\right)}{\|x\|} \cdot \vec{x}$$

Exercise: write inverse explicitly, prove it's a homeomorphism

Ex 3 $S^n - 1\,\text{pt} \simeq \mathbb{R}^n \simeq \mathring{D}^n$



Idea: stereographic projection $x \neq N \Rightarrow$ line intersects the plane $\{x_n = -1\}$ in one point $f(x)$.

Conversely, given $f(x)$ on the plane, connect it with N; line intersects S^n, not tangent $\Rightarrow$ intersects again at x

$$g(f(x)) = x.$$

Exercise: write explicit formulas for f and g and prove it's homeomorphism.

Recap: Quotient topology

X = top. space, $\sim$ equivalence relation on X

$$Y = X/\sim \;=\; \{\text{equivalence classes of } \sim\}$$

$$\pi : X \longrightarrow X/\sim \qquad \text{projection}$$
$$x \longrightarrow [x]$$

Def $U \subset X/\sim$ is open iff $\pi^{-1}(U)$ is open

$$\pi^{-1}(U) = \{x : [x] \in U\}$$

some collection of eq. classes

Exercise: $X/\sim$ is a top. space + π is a continuous map.

Warning: $X/\sim$ could be very bad (non-Hausdorff...)

$X = \mathbb{R}$ $\quad x \sim y$ if either $x \neq 0$ and $y \neq 0$

or $x = 0$ and $y = 0$.

$X/\sim = \{0, \neq 0\} = \mathbb{R}/\mathbb{R}^*$

Def $A \subset X$, we define X/A = "collapse A"

as $X/\sim$ where $x \sim y$ if either $x \in A$, $y \in A$

or $x \notin A$, $x = y$.

any two points in A are equiv.

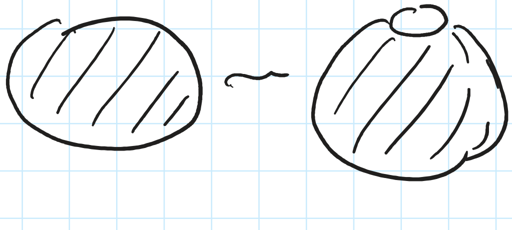
Equiv. classes $= \{a, x \notin A\}$

all pts in A

all different.

Ex $D^n / \partial D^n \underset{\text{homeo}}{=} S^n$

collapse the boundary of n-disk into one pt



Map: $D^n \setminus \partial D^n \xleftrightarrow{f} S^n - \{N\}$

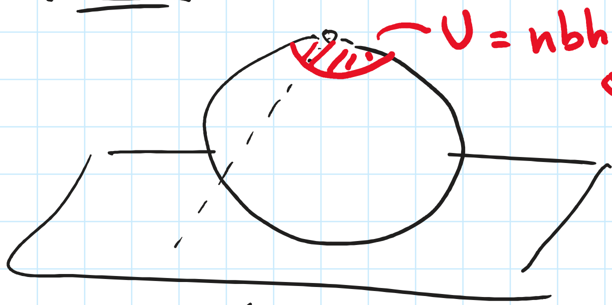
is as before

$\mathring{D}^n \simeq \mathbb{R}^n$ stereographic projection.

one equiv. class $\to \partial D^n \longrightarrow N$

Exercise: prove it's a homeomorphism with quotient topology on $D^n/\partial D^n$

<u>Idea</u>: Outside N, continuous bijection – proved already



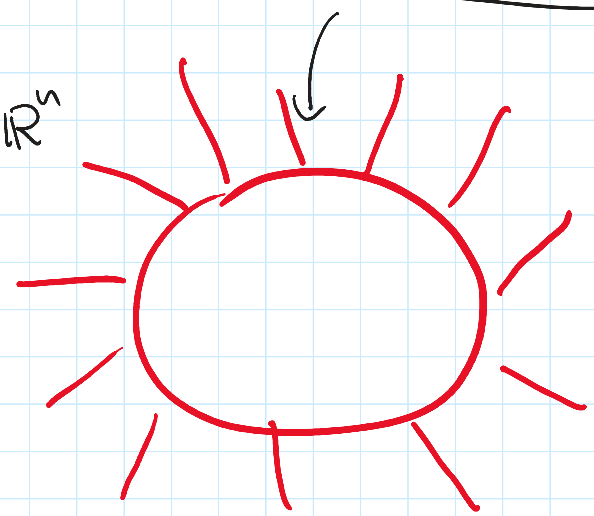
U = nbhd of N

$U - \{N\} \subset S^n \setminus \{N\}$

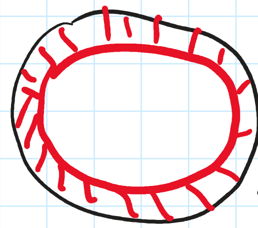
$\downarrow f$

subset of $\mathring{D}^n$

$\mathbb{R}^n$



$\mathbb{R}^n \sim \mathring{D}^n$



open subset of D^n containing ∂D^n

$D^n/\partial D^n \xleftarrow{\pi}$

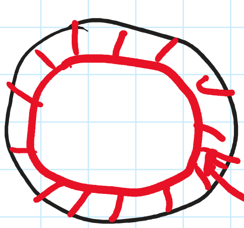
$\pi^{-1}(V)$

V = open subset of $D^n/\partial D^n$

Need to prove a bit more:

any open subset of D^n containing ∂D^n comes from this construction

<u>Hint</u>: use compactness of ∂D^n to prove that any open subset of D^n containing ∂D^n also contains



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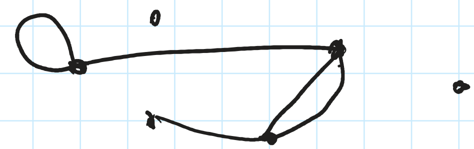
<u>Cell complexes</u>: class of top. spaces "nice"

Cell complexes. (CW complexes)

Class of top. spaces "nice"

- inductive definition X^k = k-skeleton of X

- X^0 = discrete set of points (0-cells)
- X^1 = glue in edges between the points in X^0



X^1 = a graph

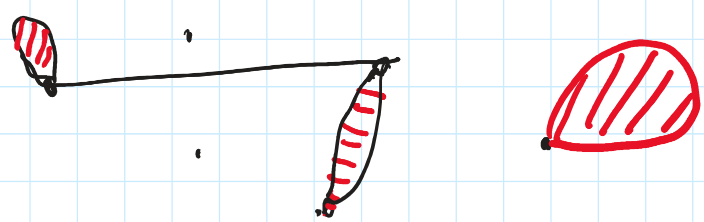
- $X^k = X^{k-1}$ with some k-cells attached along the boundary

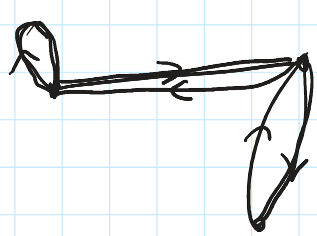
$$X^k = \left(X^{k-1} \bigsqcup_\alpha D^k_\alpha\right) / \sim$$

discs (= k-cells) D^k_α + attaching maps

$$\varphi_\alpha : \partial D^k_\alpha \longrightarrow X^{k-1}$$

$$x \sim \varphi_\alpha(x) \quad \text{for } x \in \partial D^k_\alpha .$$





Ex For any path in the graph X^1 which starts and ends at the same point (can go along edges as many times as we want).

we can get a map $S^1 \to X^1$

and use it as attaching map

$\partial D^2 \longrightarrow X^1$