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Applications of $\pi_1(S^1) = \mathbb{Z}$

Fundamental Theorem of Algebra

- If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ then $p(z)$ has a complex roots

Proof: Fix a large circle of radius R

exercise: For R sufficiently large + $|z| = R$, $|z|^n = R^n > |\varphi(z)|$

BWOC suppose $p(z) \neq 0 \ \forall z$.

$$f(z) = \frac{p(z)}{\|p(z)\|} : S^1_R \to S^1$$

Claim: $f(z)$ has degree n

Proof: $$f_t(z) = \frac{z^n + t\varphi(z)}{\|z^n + t\varphi(z)\|} : S^1_R \to S^1$$

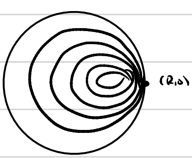
when $t = 0$ $\quad \frac{z^n}{\|z^n\|} = \left(\frac{z}{2}\right)^n$

$t = 1$ $\quad \frac{p(z)}{\|p(z)\|} = f(z)$

and well-defined since $|z|^n > |\varphi(z)| \geq |t\varphi(z)|$ since $0 \leq t \leq 1$

so $z^n + t\varphi(z) \neq 0$.

$\Rightarrow$ $f_0 \sim f_1$ homotopic $\Longleftrightarrow$ $\deg f_1 = \deg f_2 = n$



Consider a family of circles with radius $\to 0$ and fixed point $(R, 0)$

Then $g_t(z) = f(z)\big|_{\substack{\text{circle of} \\ \text{radius } tR}}$

$g_1(z) = f(z)$ and $g_0(z) =$ constant map

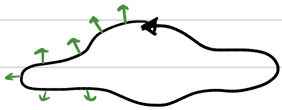
so $f(z) \sim$ constant map $\Rightarrow$ $\deg f(z) = 0$

Contradiction

Vector Fields on $\mathbb{R}^2$

$V : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

- draw a vector at every point — $P \longrightarrow V(P)$
- singular point $V = 0$



γ = smooth closed curve in $\mathbb{R}^2$

$\gamma \longleftrightarrow$ map $\gamma : S^1 \to \mathbb{R}^2$

An invariant of the vector field using γ is :

Def The index of a vector field V along γ is the degree of the map

$$\frac{V(\gamma(t))}{\|V(\gamma(t))\|} : S^1 \longrightarrow S^1$$

assume no singular points of V on γ

Properties of the Index

(1) Invariant under homotopy of γ for fixed V.

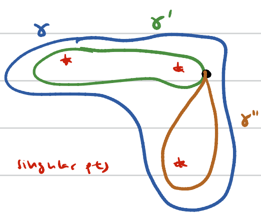
(assuming we don't cross a singular point)

- By Sard's Theorem, singular points are isolated, measure 0, and $\deg_\gamma(V) = \deg_{\gamma'}(V)$ if $\gamma \sim \gamma'$

(2) Invariant under homotopy of V for fixed γ

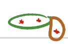
(assuming $V(\gamma(t)) \neq 0$ everywhere)

(3) Index is Additive



* singular pts

$$\deg_\gamma(V) = \deg_{\gamma'}(V) + \deg_{\gamma''}(V)$$

$\longrightarrow$ shrink γ to 

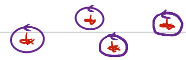
Use homotopy of γ + composition of loops in $\pi_1(S^1)$

$\deg : \pi_1(S^1) \to \mathbb{Z}$ is a group homomorphism

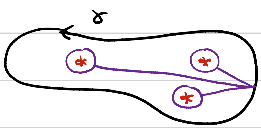
(4) Index does not depend on parametrization
(as long as you go in the same direction)

(5) If there are no singular points of v inside γ
then $\deg_\gamma(v) = 0$ (assuming γ bounds a disk)

Def The index of a singular point p of v is the index of v along a small circle around p
(oriented c.c. , radius doesn't matter)



Thm For any γ, $\deg_\gamma(v) =$ sum of indices of singular points inside γ



$\deg_\gamma(v) = 3$

Corollary

- If $\deg_\gamma(v) \neq 0$ then v has a singular point inside γ
- If v always points outside of γ then $\deg_\gamma(v) = 1$ and v must have a singular point inside γ

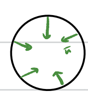
Idea : outward angle between $v(\gamma(t))$ and normal vector $\bar{n}(\gamma(t))$ is less than $\pi/2$. $V_t = Vt + \bar{n}(1-t) \neq 0$ since $(v_t, \bar{n}) > 0$

Brouwer's Fixed Point Theorem

Any continuous map $D^2 \xrightarrow{\varphi} D^2$ has a fixed point ($\exists\ p$ such that $\varphi(p) = p$)

Proof : Construct a vector field v. on $\partial D^2 = S^1$, this v.f. points inwards.
So $\deg_{\partial D^2}(v) = 1$

$\Rightarrow$ v has a singular point inside D^2. Hence $\varphi(p) = p$



degree 1
v homotopic to $\bar{n}$