Lecture 12 (2/8)

Wednesday, February 8, 2023 7:52 AM

Computations of π_1

① $\pi_1(S^n) = 0$ for $n > 1$

Proof: Use cell decomposition of S^n with one 0-cell and one n-cell

By Cellular Approximation Thm any map $\gamma : S^1 \longrightarrow S^n$ is homotopic to a cellular map (0-skeleton of S^1) $\longrightarrow$ (0-skeleton of S^n)

$$S^1 = \begin{pmatrix} \text{1-skeleton} \\ \text{of } S^1 \end{pmatrix} \longrightarrow \begin{pmatrix} \text{1-skeleton} \\ \text{of } S^n \end{pmatrix}$$

$$\begin{pmatrix} \text{0-skeleton} \\ \text{of } S^n \end{pmatrix} = \begin{pmatrix} \text{1-skeleton} \\ \text{of } S^n \end{pmatrix}$$

So any map $S^1 \longrightarrow S^n$ is homotopic to a constant map and $\pi_1(S^n) = 0 = \{e\}$

Similarly If X is a CW-complex with no 1-cells (connected) then $\pi_1(X) = 0$.

Ex $\pi_1(\mathbb{C}P^n) = 0$ for $n \geq 1$

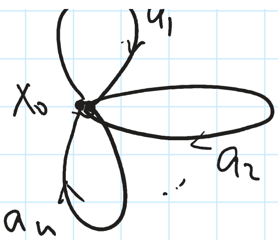
Recall: $\mathbb{C}P^n$ has one 0-cell, one 2-cell, 4-cell...

no 1-cells $\Rightarrow \pi_1 = 0$.

Ex 

$X =$ "wedge of circles"

Ex



X = "wedge of circles"
= "bouquet of circles"

$$\underbrace{S^1 \vee S^1 \vee \dots \vee S^1}_{n\ S^1\text{'s}}$$ glued at one point

One 0-cell, n 1-cells and no higher cells

Choose orientation on each S^1, and label them $a_1 \dots a_n$

Each $a_i \longleftrightarrow$ loop in $X \longleftrightarrow$ class in $\pi_1(X)$

Thm $\pi_1(X)$ = free group generated by $a_1 \dots a_n$.

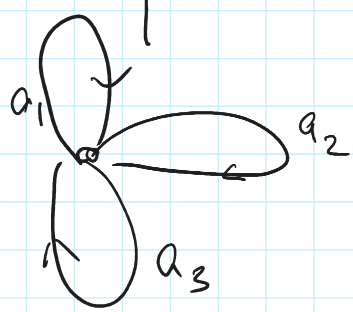
= { all words in $a_1 \dots a_n, a_1^{-1}, \dots a_n^{-1}$ }

a_i do not commute!

$a_1 a_2^{-1} a_3 a_1^{-1} a_3^{-1} a_2^{-1} a_1^5$

$a_1 a_1^{-1} = e$

a_i^{-1} = same loop as a_i with opposite orientation



$a_1 a_2 a_1^{-1} a_2^{-1} \neq e$

Ex

$n=1$ $\pi_1(S^1)$ = free group with 1 generator

$\langle a_1 \rangle = \{ a_1^m, m \in \mathbb{Z} \} \simeq \mathbb{Z}$

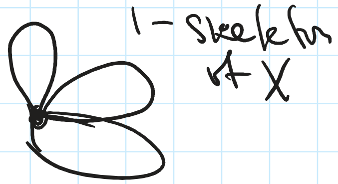
Cor For any (connected) graph G with V vertices and E edges, $\pi_1(G)$ = free group with

$E-V+1$ generators.

Proof (HW: G is homotopy equivalent to a graph with 1 vertex and $E-V+1$ loops).

Thm Suppose X is a CW complex with one 0-cell. Then $\pi_1(X)$ has the following presentation:

- Generators $\longleftrightarrow$ 1-cells
- Relations $\longleftrightarrow$ 2-cells



And all higher cells do not affect $\pi_1(X)$.

Relations: Given a 2-cell, there is an attaching map

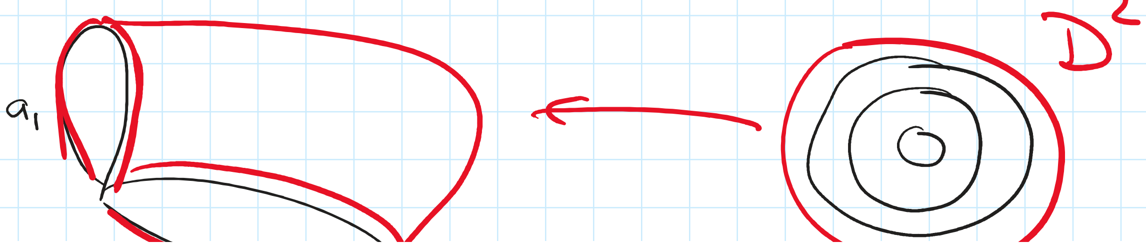
$$\phi_i : \underset{\underset{S^1}{\|}}{\partial D^2} \longrightarrow \left(\begin{matrix}\text{1-skeleton}\\ \text{of } X\end{matrix}\right)$$

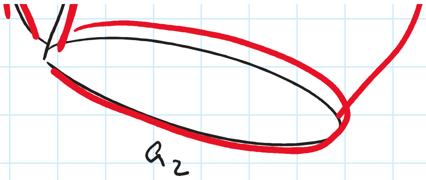
$\phi_i(S^1)$ = loop in 1-skeleton of X

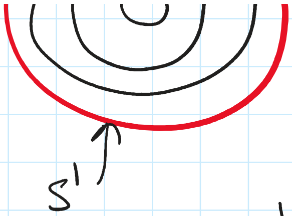
homotopic to a word in $a_1 \ldots a_n$

Relation: $\phi_i(S^1) = \left(\begin{matrix}\text{or corresponding}\\ \text{word in } a_i\end{matrix}\right) = e$

So that this loop $\phi_i(S^1)$ is contractible in X

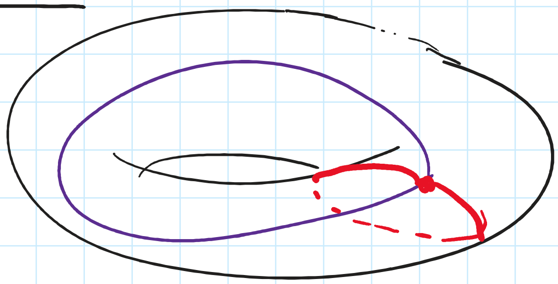


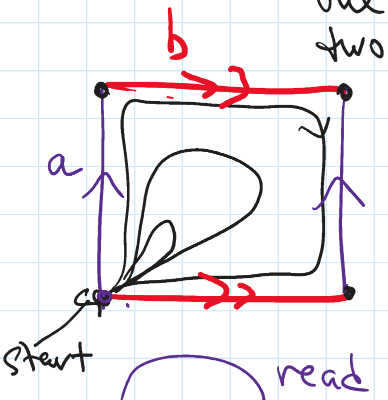




We can shrink the loop $\phi_i(S^1)$ using concentric circles in D^2 ⇒ contractible inside 2-cell ⇒ contractible in X.

Examples T^2





One 0-cell
two 1-cells
one 2-cell

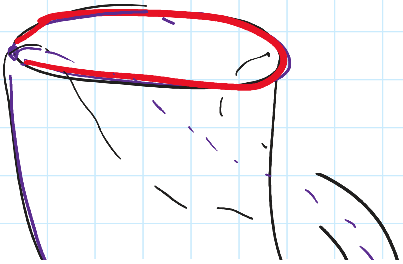
Generators: a, b

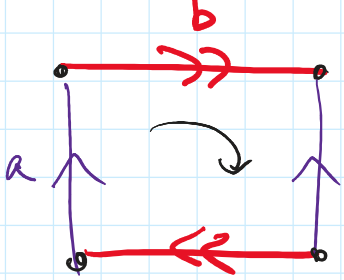
Relations: $aba^{-1}b^{-1} = e$

Can start at a different vertex of square: $ba^{-1}b^{-1}a = e$ (equivalent relation!)

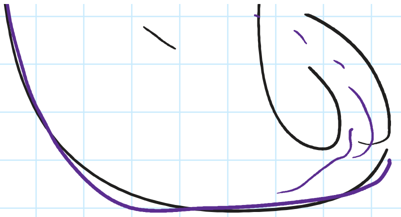
Conclusion: $\pi_1(T^2) = \dfrac{\langle a,b\rangle}{(aba^{-1}b^{-1}=e)} = \dfrac{\langle a,b\rangle}{(ab=ba)} = \mathbb{Z}^2$

Ex Klein bottle





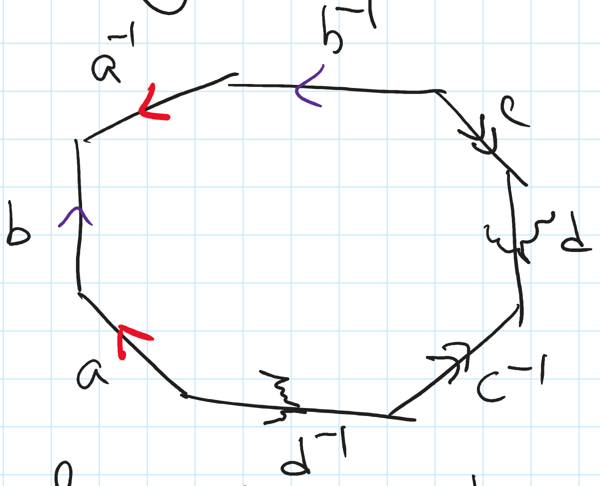
$\pi_1\left(\text{Klein bottle}\right) = \dfrac{\langle a,b\rangle}{\langle aba^{-1}b=e\rangle}$



$\langle a b a^{-1} b = e \rangle$

Ex Genus g orientable surface

g=2



one 0-cell, 4 1-cells, 1 2-cell

$$\pi_1(X) = \frac{\langle a,b,c,d \rangle}{\langle aba^{-1}b^{-1}cdc^{-1}d^{-1} = e \rangle}$$

In general:

$$\frac{\langle a_1, b_1, a_2, b_2, \ldots a_g, b_g \rangle}{\langle a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots \_\_ - a_g b_g a_g^{-1} b_g^{-1} = e \rangle}$$

Rmk X = finite CW complex

$\Rightarrow$ $\pi_1(X)$ is finitely presented

(finitely many generators, finitely many rels)

Conversely, any finitely presented group appears as $\pi_1(X)$!

1-cell for each generator

2-cell for each relation, attached along the corresponding word in generators.