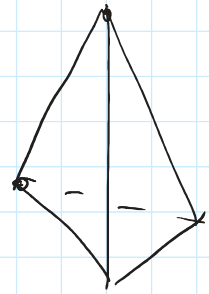
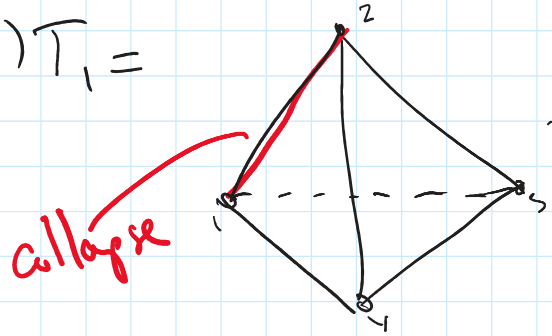
Lecture 13 (2/10)

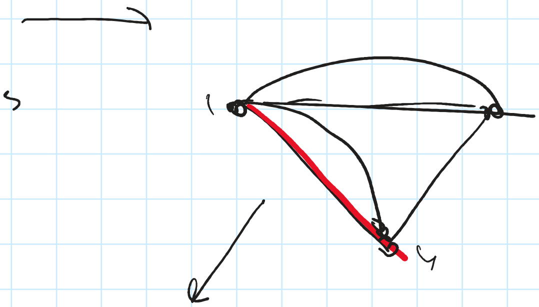
Friday, February 10, 2023 9:01 AM

(1) Let T be the 3-dimensional solid tetrahedron, and T_k its k-skeleton. In other words, T_1 is the union of vertices and edges of T, T_2 is the union of vertices, edges, and faces, and $T_3 = T$.
  (a) Find the fundamental group of T_1.
  (b) Find the fundamental group of T_2.
  (c) Find the fundamental group of T_3.

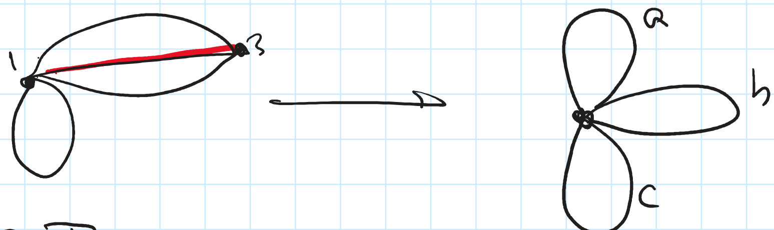


(a) $T_1 =$



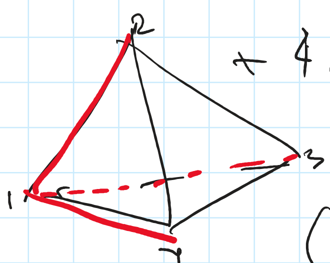


Collapsing a contractible subcomplex is a homotopy equivalence $\Rightarrow$ does not change π_1



π_1 = free group generated by a, b, c.

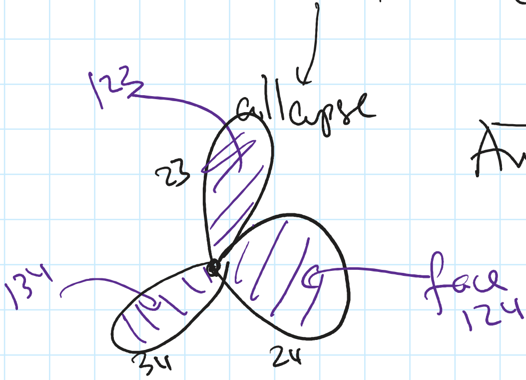
(b) T_2



+ 4 faces $T_2 \simeq S^2$

$\pi_1(T_2) = \pi_1(S^2) = \{e\}$

(inscribe T into S^2 and project from the center to S^2)



Another solution

$$\pi_1(T_2) = \frac{\langle a, b, c\rangle}{\langle a=e, b=e, c=e,\ abc=e\rangle} = \{e\}$$

+ face 234.

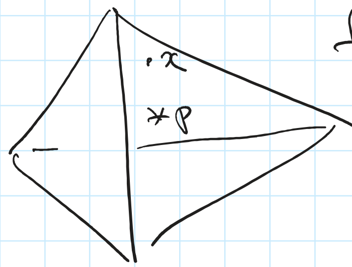
34 24 $\langle a=e, b=e, c=e$

+ face 234. $abc = e \rangle$ 234

(c) T_3 : add 3-cell

contractible

$\Rightarrow \pi_1(T_3) = \{e\}$



$f_t(x) = tx + (1-t)p$

Contained in T

since T convex

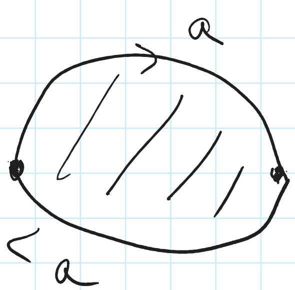
$t=0 \quad f_0(x) = p \quad t=1 \quad f_1(x) = x$

Another solution:

attaching 3-cells does not change $\pi_1 \Rightarrow$

$\pi_1(T_2) = \pi_1(T_3) = \{e\}$.

<u>Ex</u> $\pi_1(\mathbb{RP}^2) = \langle a \rangle / \langle a \cdot a = e \rangle = \dfrac{\langle a \rangle}{\langle a^2 = e \rangle} = \mathbb{Z}_2$



one 0-cell

one 1-cell

one 2-cell

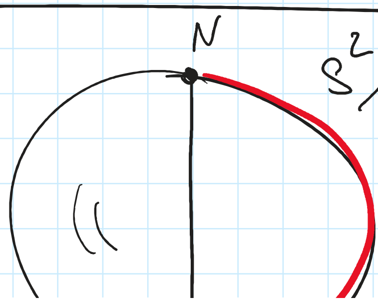
$\boxed{\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2}$

$\pi_1(\mathbb{RP}^n) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ for all $n \geq 2$

since adding 3, 4, ... - cells

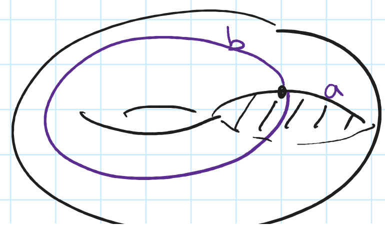
does not change π_1.

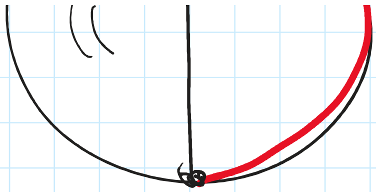
<u>Ex</u>

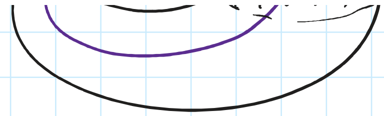


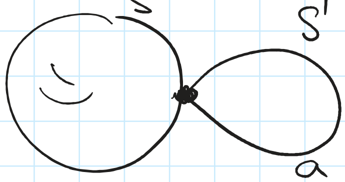
$S^2/N\sim S$

$\sim$









one 0-cell
one 1-cell
one 2-cell

$$\frac{\langle a \rangle}{\langle e = e \rangle} = \mathbb{Z}$$

no relation

one 0-cell
two 1-cells a, b
two 2-cells (2-cell in T^2 + disk)

$$\frac{\langle a, b \rangle}{\langle aba^{-1}b^{-1} = e \ \& \ a = e \rangle} = \mathbb{Z}\langle b \rangle$$

(last time)

Seifert - Van Kampen Theorem

$X = U_1 \cup U_2$, assume that $U_1 \cap U_2$ is <u>path-connected</u>.

open

Then $\pi_1(X)$ is amalgamated free product of $\pi_1(U_1)$ and $\pi_1(U_2)$ along $\pi_1(U_1 \cap U_2)$

$$\begin{tikzcd} & U_1 \\ U_1 \cap U_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & \\ & U_2 \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(U_1) \\ \pi_1(U_1 \cap U_2) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & \\ & \pi_1(U_2) \end{tikzcd}$$

inclusion

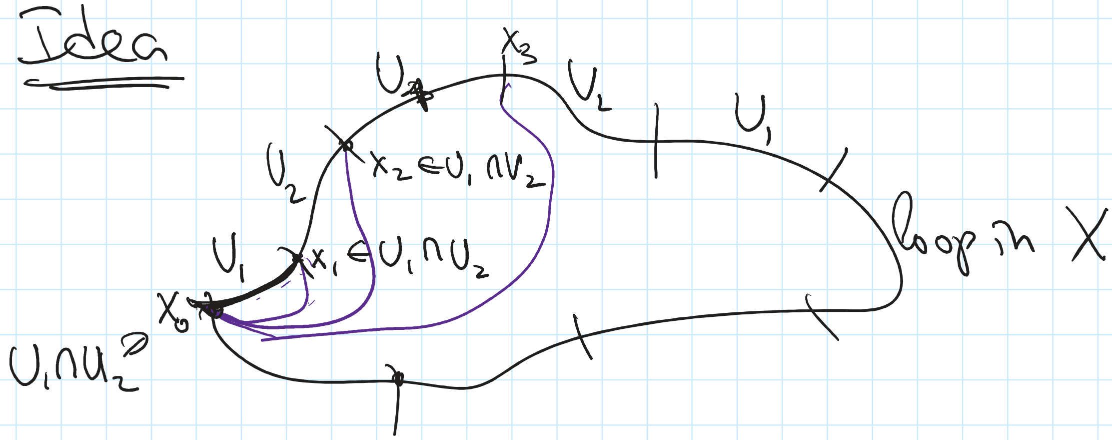
$\pi_1(X)$ is freely generated by $\pi_1(U_1)$ and $\pi_1(U_2)$ modulo relations $i_{1*}(\gamma) = i_{2*}(\gamma)$ for all $\gamma \in \pi_1(U_1 \cap U_2)$

free product

$$= \frac{\pi_1(U_1) * \pi_1(U_2)}{\langle i_{1*}(\gamma) = i_{2*}(\gamma),\ \gamma \in \pi_1(U_1 \cap U_2)\rangle}$$

$$= \frac{\text{free}\langle \text{all generators of } \pi_1(U_1), \text{ all generators of } \pi_1(U_2)\rangle}{\langle \text{relations from } \pi_1(U_1),\ \text{relations from } \pi_1(U_2),\ i_{1*}(\gamma) = i_{2*}(\gamma)\rangle}$$

Idea



Since $U_1 \cap U_2$ is path-connected, we can connect x_i to x_0 by paths (blue)

$$\begin{pmatrix}\text{original}\\ \text{loop}\end{pmatrix} = \left(\text{loop in } U_1\right) * \left(\text{loop in } U_2\right) * \left(\text{loop in } U_1\right) * \cdots$$

$\Rightarrow \pi_1(X)$ is generated by $\pi_1(U_1)$ and $\pi_1(U_2)$.

Application 1 If $\pi_1(U_1) = \pi_1(U_2) = \{e\}$

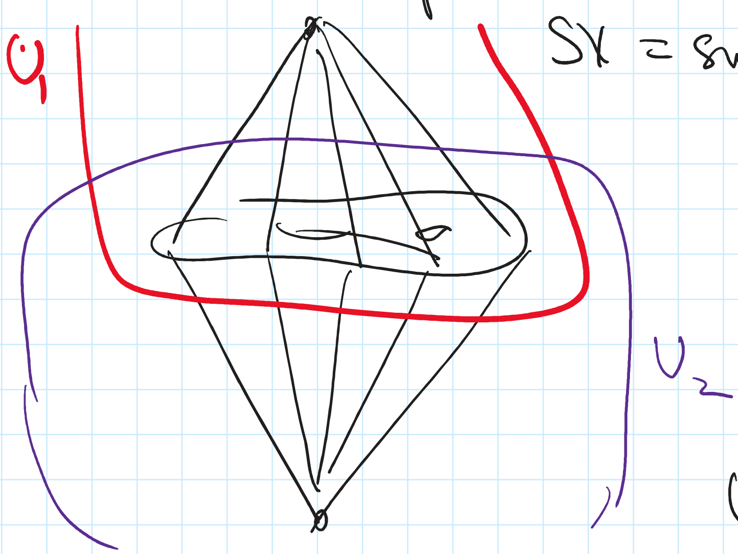
and $U_1 \cap U_2$ path connected

then $\pi_1(X) = \{e\}$

($\Rightarrow \pi_1(S^n)$ is trivial for $n \geq 2$)

Ex If X is path-connected then $\pi_1(SX) = \{e\}$

SX = suspension of X.



$U_i \sim$ cone (X) is contractible

$\Rightarrow \pi_1(U_1) = \pi_1(U_2) = \{e\}$

$U_1 \cap U_2 = X \times (-\varepsilon, \varepsilon)$ is path-connected.

$\Rightarrow$ by Seifert-Van Kampen $\pi_1(SX) = \{e\}$.

$SX = X \times [-1, 1] / \sim$ $\qquad U_1 = X \times (-\varepsilon, 1] / \sim$

$U_2 = X \times [-1, \varepsilon) / \sim$

Abelianization $G \twoheadrightarrow G / [G, G]$

noncomm $\qquad$ commutative.

$\underline{\underline{\text{Ex}}}$ Σ = genus g surface

$$\pi_1(\Sigma) = \frac{\langle a_1, b_1, \ldots a_g, b_g \rangle}{\langle \underbrace{a_1 b_1 a_1^{-1} b_1^{-1}} \ldots \underbrace{a_g b_g a_g^{-1} b_g^{-1}} = e \rangle}$$

$\Big\downarrow$ abelianization

$$\pi_1(\Sigma)^{ab} = \frac{\mathbb{Z}\langle a_1, b_1, \ldots a_g, b_g \rangle}{\langle e \rangle} = \mathbb{Z}^{2g}$$

comm.