Heegard decompositions

Σ_g = genus g surface

M_1, M_2 = genus g handlebodies with boundary on Σ

handlebody = inside of Σ_g in $\mathbb{R}^3$

$\varphi : \Sigma \to \Sigma$ homeomorphism

M = gluing of $M_1 \cup M_2$ along (Σ, φ)

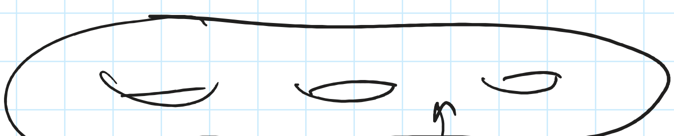
$= M_1 \sqcup M_2 \big/ p \sim \varphi(p)$ ← gluing

$p \in \Sigma \subset M_1 \qquad \varphi(p) \in \Sigma \subset M_2$.

__Thm__ 1) If φ is smooth, then M is a smooth 3-manifold

2) __Any__ closed oriented 3-manifold can be obtained by this construction.

__Ex__ $S^3 = M = M_1 \cup M_2$



← outside $= M_2$ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$

inside $= M_1$

__Exercise__ M_2 is also a genus g handlebody!

__Q__ How to use this to compute $\pi_1(M)$? | for any g.

Use Seifert–van Kampen Thm:

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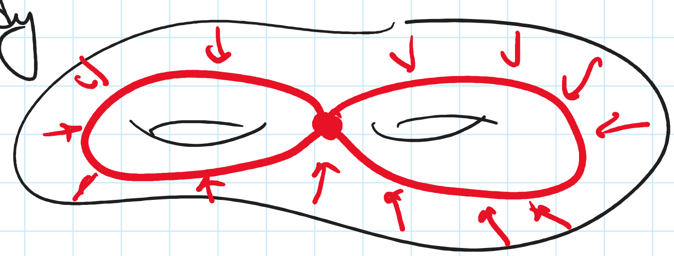
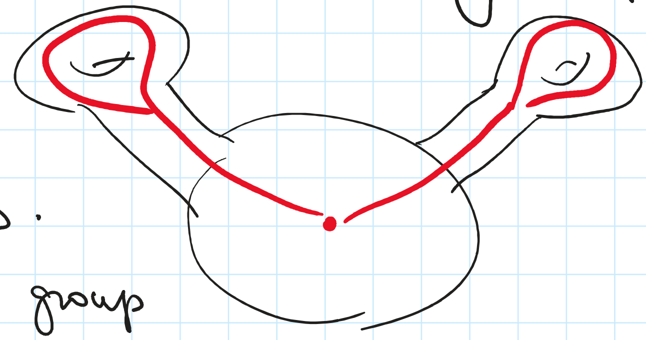
U_i = open subset "thickening" M_i

$= M_i \cup$ small neighborhood of Σ.

$\pi_1(U_1) = \pi_1(M_1)$, need to understand π_1 for a handlebody



We can deformation retract U_1 or M_1 onto a graph with 1 vertex and g loops.



$\Rightarrow \pi_1(M_1)$ = free group with g generators.

Similarly, $\pi_1(M_2)$ = free group with g generators.

$U_1 \cap U_2$ = neighborhood of Σ_1 in M

$\Rightarrow U_1 \cap U_2 \sim \Sigma_1 \Rightarrow \pi_1(U_1 \cap U_2) = \pi_1(\Sigma)$ <u>know</u>

2g generators, 1 relation

Need to understand inclusion maps

$$\begin{tikzcd} & \pi_1(U_1)=\pi_1(M_1) \\ \pi_1(U_1 \cap U_2) \arrow[ur, "j_1"] \arrow[dr, "j_2"'] \arrow[d, equal] & \\ \pi_1(\Sigma) & \pi_1(U_2)=\pi_1(M_2) \end{tikzcd}$$

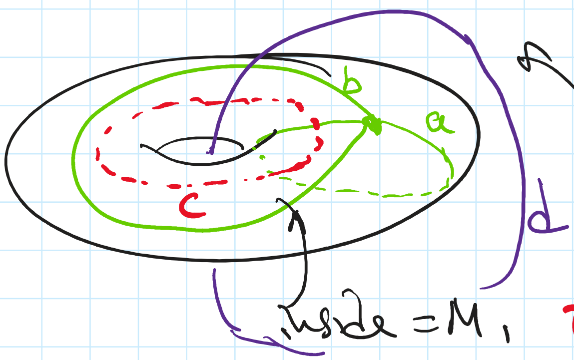
<u>Seifert–van Kampen</u>: $\pi_1(M)$ = amalgamated free product of $\pi_1(M_1)$

Seifert–van Kampen: $\pi_1(M) =$ amalgamated free product of $\pi_1(U_1)$ and $\pi_1(U_2)$ along j_1, j_2

$$= \frac{\left\langle \text{generators: } \begin{matrix} g \text{ generators of } \pi_1(U_1) \\ g \text{ generators of } \pi_1(U_2) \end{matrix} \right\rangle}{\left\langle j_1(\gamma) = j_2(\gamma) \text{ for any generator } \gamma \text{ of } \pi_1(U_1 \cap U_2) \right\rangle}$$

In general, need to add relations in $\pi_1(U_i)$.

Ex $S^3 = M_1 \underset{T^2}{\cup} M_2$ $\qquad \pi_1(\Sigma) = \dfrac{\langle a, b \rangle}{\langle aba^{-1}b^{-1} = e \rangle} = \dfrac{\langle a, b \rangle}{ab = ba}$



outside $= M_2$ $\qquad \pi_1(M_2) = \langle d \rangle$

inside $= M_1$ $\qquad \pi_1(M_1) = \langle c \rangle$

$j_1 : \pi_1(\Sigma) \longrightarrow \pi_1(M_1)$

$a \longrightarrow e$ (can shrink a to a point inside M_1)

$b \longrightarrow c$

$j_2 : \pi_1(\Sigma) \longrightarrow \pi_1(M_2)$

$a \longrightarrow d$

$b \longrightarrow e$ (can shrink b to a point in M_2)

Seifert-van Kampen: $\pi_1(S^3) = \dfrac{\overbrace{\langle c, d \rangle}^{\text{generators of } \pi_1(M_1) \text{ and } \pi_1(M_2)}}{\langle j_1(a) = j_2(a) \rangle}$

Kampen

$$\frac{\quad}{\left\langle \begin{matrix} j_1(a) = j_2(a) \\ j_1(b) = j_2(b) \end{matrix} \right\rangle}$$

$$= \frac{\langle c, d \rangle}{\left\langle \begin{matrix} e = d \\ c = e \end{matrix} \right\rangle} = \{e\}.$$

Q: What is homeomorphism $\varphi : T^2 \to T^2$ in this case?

Q: What if we choose a different φ?

Q: Why $M_2 = S^3 \setminus M_1$ is a solid torus?

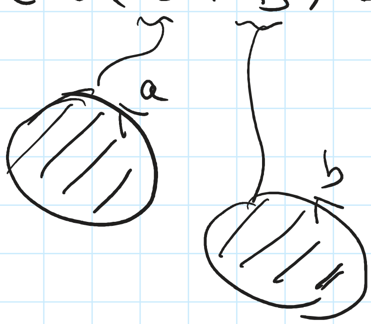
$S^3 = \partial B^4$ in $\mathbb{R}^4$

$B^4 \approx B^2 \times B^2$

convex body in $\mathbb{R}^4$

$\partial B^4 \approx \partial(B^2 \times B^2) = \partial B^2 \times B^2 \cup B^2 \times \partial B^2$

$= (S^1 \times B^2) \cup (B^2 \times S^1)$



solid torus
$a \times B^2$
$\cup$
"core" $= a \times \{0\}$
B^2 retracts to $\{0\}$
so $a \times B^2$ retracts to $a \times \{0\} = d$

solid torus
$B^2 \times b$
$\cup$
"core" $= \{0\} \times b$
$B^2 \times b$ retracts to $\{0\} \times b = c$

Intersection of solid tori $= S^1 \times S^1 = a \times b$

Note: the homeomorphism $M_1 \approx M_2$

swaps a and b on the boundary.

<u>Rmk</u> $S^1 = \mathbb{R}/\mathbb{Z}$

$T^2 = S^1 \times S^1 = (\mathbb{R}/\mathbb{Z})^2 = \mathbb{R}^2/\mathbb{Z}^2 = \{(x,y) \in \mathbb{R}^2\} / \begin{matrix}(x+2\pi k \\ y+2\pi l) \\ \sim (x,y)\end{matrix}$

Given a matrix $A \in SL(2,\mathbb{Z})$
we can define a homeomorphism of the torus

$$\varphi_A(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{since } A\mathbb{Z}^2 = \mathbb{Z}^2$$

Can use it to construct lots of interesting spaces by gluing $M_1 \cup M_2$ along (T^2, φ_A).

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } S^3$$ (swap a and b up to change of orientation)

<u>Def:</u> A lens space is the result of gluing of $M_1 \cup M_2$ along some φ_A

<u>HW</u> Compute π_1(result) in terms of A.

$$\pi_1(\text{result}) = \frac{\langle c, d \rangle}{\langle \text{relations} = ? \rangle}$$