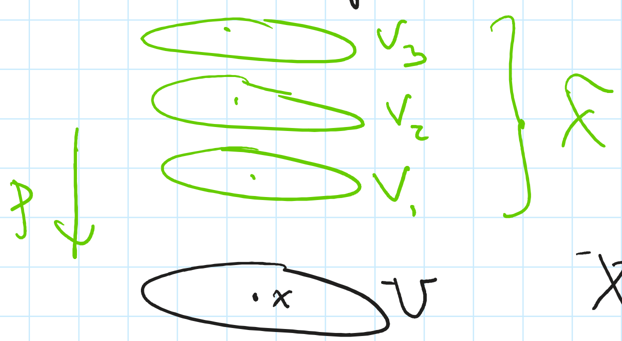
Covering spaces

Def $p: \tilde{X} \longrightarrow X$ is a covering map if for every point $x \in X$ there is a neighborhood $U \ni x$ such that $p^{-1}(U) = \bigsqcup V_i$

V_i open in $\tilde{X}$, V_i homeomorphic to U via p

disjoint in $\tilde{X}$



$\Rightarrow p^{-1}(x)$ is a discrete subset of $\tilde{X}$,

one point in each V_i

Ex 1 $\tilde{X} = \mathbb{R}$ $X = S^1$

$$p(\varphi) = (\cos\varphi, \sin\varphi)$$

Ex 2 $\tilde{X} = S^1$ $X = S^1$

degree n map

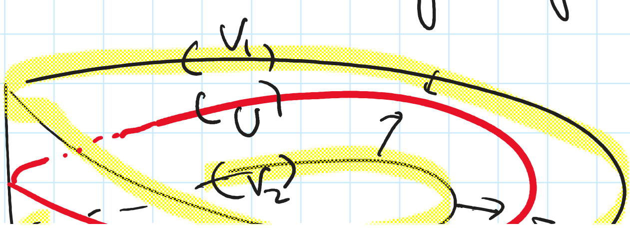
$$p(\cos(\varphi), \sin\varphi) = (\cos(n\varphi), \sin(n\varphi))$$

$$\varphi \to \varphi + 2\pi \longrightarrow n\varphi \mapsto n\varphi + 2n\pi \sim n\varphi$$

Degree n covering map.

Every point has exactly n preimages

Ex 3



boundary of Mobius band projects to

band projects to the central circle

This is a degree 2 covering map $S^1 \to S^1$

Ex 4 $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$S^1 \times S^1 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$$

$p: \mathbb{R}^2 \longrightarrow T^2$ covering map of infinite degree.

$\sim$ square grid

$\leftarrow p^{-1}(U)$

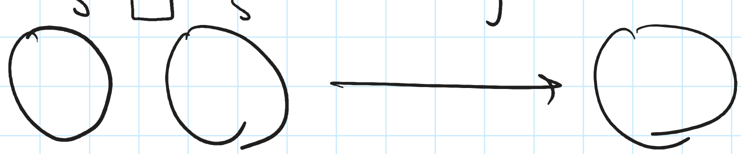
Ex 5 $p: S^n \longrightarrow \mathbb{R}P^n = S^n/\pm 1$

Covering map of degree 2.

Ex 6 $\tilde{X} = X \sqcup X \xrightarrow{(id,\, id)} X$

clearly a covering map of degree 2

$S^1 \sqcup S^1$



S^1

More generally, disjoint union of coverings is a covering.

Properties ① If X is connected $\tilde{X} \to X$ covering map then all points in X

covering map

then all points in X have same number of preimages.

<u>Idea of proof</u> For every k, $\{x \in X : p^{-1}(x) \text{ has } k \text{ points}\}$ open in X

If for two different k these are nonempty $\Rightarrow X$ is union of disjoint opens, contradiction. ∎

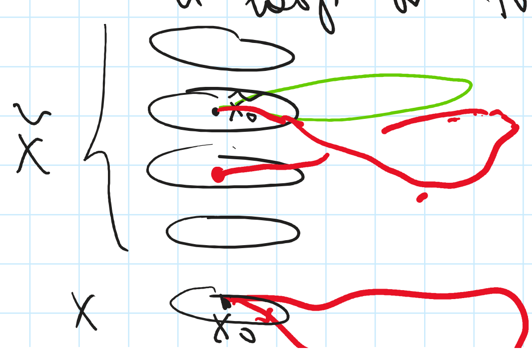
<u>Thm</u> (Hatcher 1.30) $p : \tilde{X} \to X$ covering map
$\tilde{x}_0 \to x_0$

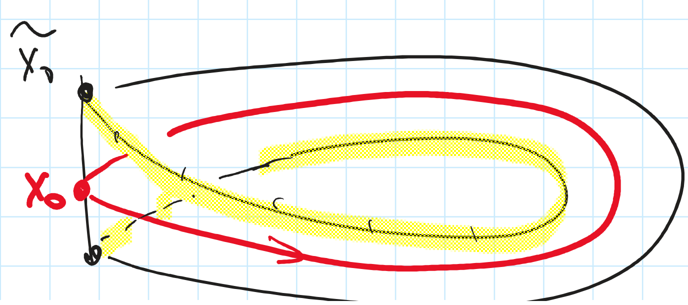
Choose $\tilde{x}_0$ such that $p(\tilde{x}_0) = x_0$

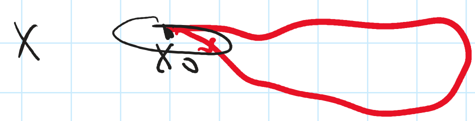
(a) Any continuous path in X starting at x_0 lifts to a unique path in $\tilde{X}$ starting at $\tilde{x}_0$

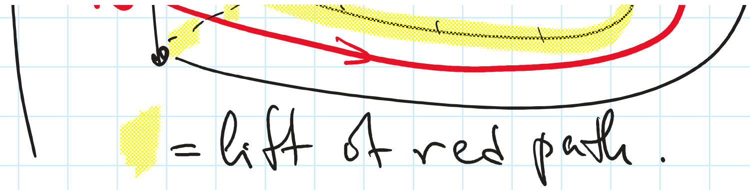
(b) Any homotopy of a path in X lifts to a homotopy of paths in $\tilde{X}$.

<u>Warning:</u> A closed loop in X can either lift to a loop or to a path in $\tilde{X}$!









= lift of red path.

Thm (1.31–1.32) $p: \tilde{X} \longrightarrow X$ covering map

(a) $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ group homomorphism.
is injective

(b) Image of $p_* = \{$ loops in (X, x_0) which lift to loops in $(\tilde{X}, \tilde{x}_0)$ $\} / \sim$

(c) If $\tilde{X}$ is path connected, then

$\left(\begin{array}{c}\text{degree of}\\ \text{covering}\end{array}\right)$ = index of $p_*\pi_1(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$
(# cosets)

Proof: (a) Enough to show $\operatorname{Ker} p_* = \{e\}$.

By definition, $\operatorname{Ker} p_* = \{ \gamma$ loop in $(\tilde{X}, \tilde{x}_0)$ such that $p(\gamma)$ is homotopic to a constant loop in X $\}$.

h_t = homotopy contracting $p(\gamma)$

By Thm above, we can lift h_t to a homotopy $\tilde{h}_t$ of loops in $\tilde{X}$:

$\tilde{h}_0 = \gamma \qquad p(\tilde{h}_1) = h_1 = \{x_0\}$

$\Rightarrow \tilde{h}_1$ is contained in $p^{-1}(x_0)$ = discrete set

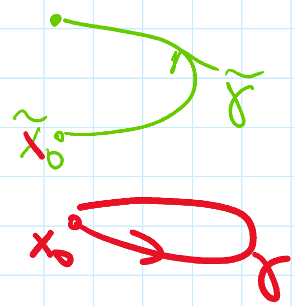
$\Rightarrow \tilde{h}_1$ is constant too.

(b) clear, by lifting property two loops are homotopic in $X \Longleftrightarrow$ their lifts are homotopic in $\tilde{X}$.

(c) We want to construct a bijection

$$\left\{ \begin{matrix} \text{sheets} \\ \text{of } p \end{matrix} \right\} \longleftrightarrow p^{-1}(x_0) \longleftrightarrow \text{cosets in } \frac{\pi_1(X)}{p_* \pi_1(\tilde{X})}$$

$H = p_* \pi_1(\tilde{X})$



$\gamma \in \pi_1(X) \longrightarrow$ lift $\tilde{\gamma}$ = path in $\tilde{X}$

endpoint $\tilde{\gamma}(1) \in p^{-1}(x_0)$

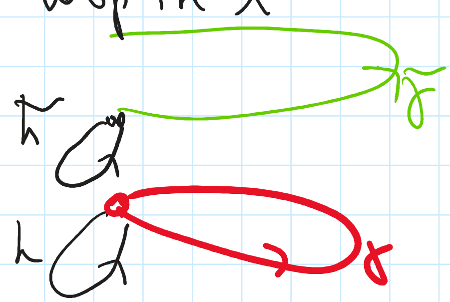
homotopy of γ_t $\xrightarrow[\text{Thm}]{\text{by}}$ homotopy of $\tilde{\gamma}_t$

$\tilde{\gamma}_t(1)$ is continuous and contained in $p^{-1}(x_0) \Rightarrow$ constant

Suppose $h \in H$, so h lifts to a loop in $\tilde{X}$

$h\gamma \xrightarrow{\text{lift}} \tilde{h}\,\tilde{\gamma} = (\widetilde{h\gamma})$

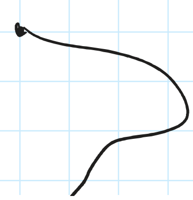
$\tilde{h}$ starts and ends at $x_0 \Rightarrow$



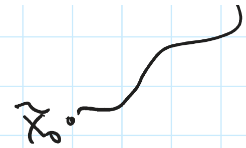
$\Rightarrow \tilde{h}\tilde{\gamma}$ has same endpoint as $\tilde{\gamma}$

Altogether, this gives a map

$$\begin{pmatrix} \text{cosets} \\ \gamma \sim h\gamma \end{pmatrix} \xrightarrow{\ \Phi\ } p^{-1}(x_0)$$



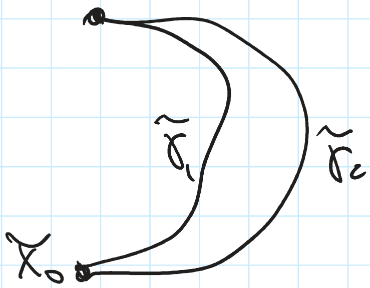
$\left(\begin{matrix}\text{cosets} \\ \gamma \sim h\gamma \\ h \in H\end{matrix}\right) \xrightarrow{\Phi} p^{-1}(x_0)$



$\tilde{X}$ path connected $\longrightarrow$ Φ surjective

$\Rightarrow$ can connect $\tilde{x}_0$ to any pt in $p^{-1}(x_0)$ by path

Φ injective:



$\tilde{\gamma}_1, \tilde{\gamma}_2$ end at same point

$\Rightarrow \tilde{\gamma}_1 \tilde{\gamma}_2^{-1}$ is a loop in $\tilde{X}$

$\Rightarrow$ element in h

$\gamma_1 = h\gamma_2$ since $\tilde{\gamma}_1 = (\tilde{\gamma}_1 \tilde{\gamma}_2^{-1})\tilde{\gamma}_2$.