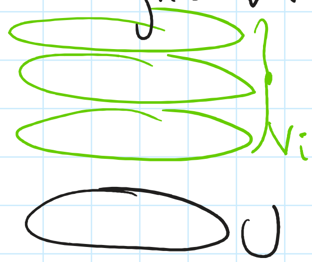
Lecture 16 (2/17)

Friday, February 17, 2023 9:53 AM

Coverings $p: \tilde{X}, \tilde{x}_0 \longrightarrow X, x_0$

Covering map: for all $x \in X$ there is a neighborhood $U \ni x$ such that $p^{-1}(U) = \sqcup V_i$

V_i all homeo to U
disjoint



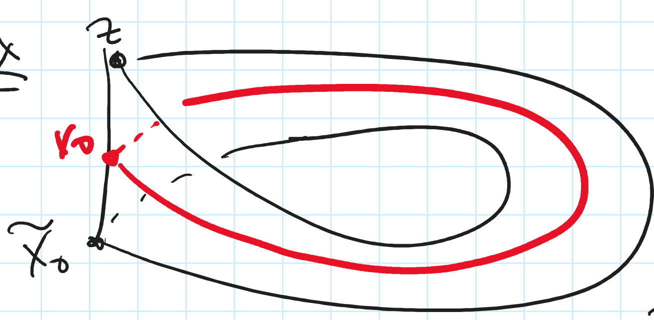
Thm If $p: (\tilde{X}, \tilde{x}) \longrightarrow (X, x_0)$ is a covering map then

(a) $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is injective

(b) $H = p_* \pi_1(\tilde{X}, \tilde{x}_i)$ consists of loops in X which lift to loops in $\tilde{X}$

(c) If $\tilde{X}$ is connected then there is a bijection $\{\text{sheets of the cover}\} \leftrightarrow \boxed{p^{-1}(x_0)} \longleftrightarrow H \backslash \pi_1(X, x_0)$ $\boxed{\text{cosets}}$

Ex



$\tilde{X} =$ (Mobius band) $= S^1$

$\big\downarrow$ 2:1

$X = S^1 =$ central circle

$\pi_1(\tilde{X}) = \mathbb{Z} = \pi_1(X)$

$p_*: \overset{\mathbb{Z}}{\overset{\shortparallel}{\pi_1(\tilde{X}, \tilde{x}_0)}} \longrightarrow \pi_1(X, x_0) = \mathbb{Z}$

$$a\cdot[\text{gen}] \longrightarrow 2a\cdot[\text{gen}]$$

$$p_*: \mathbb{Z} \xrightarrow{2} \mathbb{Z} \qquad \underline{\text{injective}}!$$

$$H = p_*\pi_1(\tilde{X}) = \text{image of } p_* = \{\text{all even integers}\} < \mathbb{Z}$$

$$\{\text{all even integers}\} = 2\mathbb{Z}$$

Thm (c): cosets $\pi_1(X)/H = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$

two cosets {even integers} {odd integers}

{even integers}: paths in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_0$ (= loops)

{odd integers}: paths in $\tilde{X}$ from $\tilde{x}_0$ to z (not loops!)

$p^{-1}(x_0) = \{\tilde{x}_0, z\}$.

<u>Ex</u> $p: \mathbb{R} \longrightarrow S^1$

$\varphi \longrightarrow (\cos\varphi, \sin\varphi)$

$\pi_1(\mathbb{R}) = 0$, $\pi_1(S^1) = \mathbb{Z}$, p_* injective

$$p^{-1}(x_0) \longleftrightarrow \text{cosets } \frac{\pi_1(S^1)}{\pi_1(\mathbb{R})} = \pi_1(S^1)$$

$$p^{-1}(x_0) = \mathbb{Z}$$

This gives a different proof (using Thm) that $\pi_1(S^1) = \mathbb{Z}$.

<u>Def</u> $p: \tilde{X} \longrightarrow X$ is a <u>universal covering</u> if $\tilde{X}$ is connected and $\pi_1(\tilde{X}) = \{e\}$.

Corollary $p: \tilde{X} \to X$ and $\pi_1(\tilde{X}) = \{e\}$
$\tilde{X}$ connected

$$p^{-1}(x_0) \underset{\text{bijection}}{\longleftrightarrow} \text{cosets } \frac{\pi_1(X)}{p_*\pi_1(\tilde{X})} = \pi_1(X)$$

$\boxed{\text{This is an extremely powerful tool for computing } \pi_1(X)\text{!}}$

Ex $S^2 \xrightarrow[2:1]{} \mathbb{R}P^2 = S^2/_{\pm 1}$

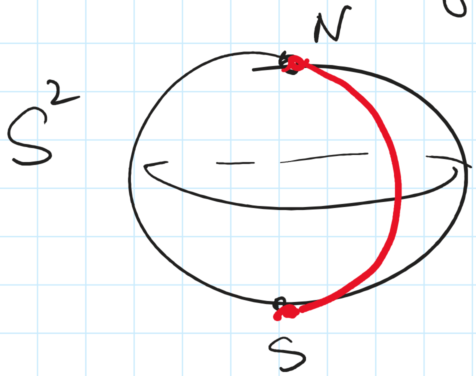
$\tilde{X} = S^2$, $\quad X = \mathbb{R}P^2$

$\pi_1(S^2) = \{e\}$
S^2 connected $\Rightarrow$ this is a universal cover

$$p^{-1}(x_0) = \{2 \text{ points}\} \overset{\text{bij}}{=} \pi_1(\mathbb{R}P^2)$$

So $\pi_1(\mathbb{R}P^2)$ is a group with 2 elements $\Rightarrow \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$.

Can visualize a generator of $\pi_1(\mathbb{R}P^2) \longleftrightarrow$ nontrivial coset



any loop in $\mathbb{R}P^2$ which does not lift to a loop in S^2.

Any path connecting N and S would do!

- Since $N \sim S$, this projects to a loop in $\mathbb{R}P^2$

- By uniqueness of lifts, this loops lifts to a path which is not a loop $\Rightarrow$ nontrivial in $\pi_1(\mathbb{R}P^2)$!

__Exercise__: Same for $p: S^n \xrightarrow[2:1]{} \mathbb{R}P^n$

similarly, $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2 \quad n \geq 2$

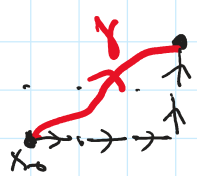
__Ex__ $p: \mathbb{R}^2 \longrightarrow T^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$(x,y) \sim (x,y) + (n_1, n_2)$$

where $(n_1, n_2) \in \mathbb{Z}^2$

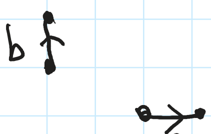
$\pi_1(\mathbb{R}^2) = \{e\}$, $\mathbb{R}^2$ connected $\Rightarrow \mathbb{R}^2$ is the universal cover of T^2.

$\Rightarrow$ bijection $\pi_1(T^2) \longleftrightarrow p^{-1}(x_0) = \mathbb{Z}^2$



A loop on $T^2 \longleftrightarrow$ class in $\pi_1(T^2)$

path in $\mathbb{R}^2$ connecting two equivalent points.



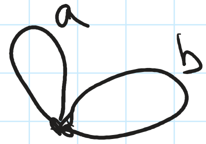
Loop in T^2

$$\pi_1(T^2) = \mathbb{Z}^2 = \frac{\langle a, b \rangle}{\langle ab = ba \rangle}$$

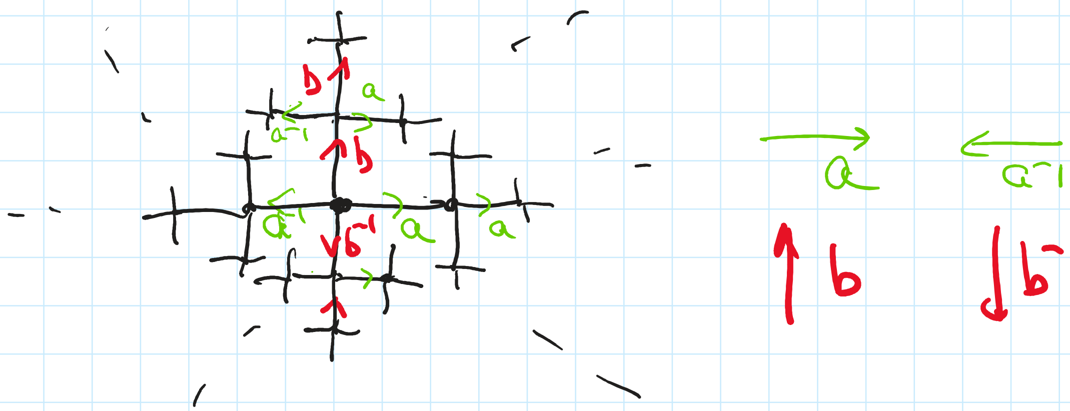
γ and $a^3 b^2$ have the same endpoints $\Rightarrow$ represent the same element in $p^{-1}(x_0)$ //

same element in $\pi_1(T^2)$

So $\boxed{\gamma = a^3 b^2 \text{ in } \pi_1(T^2)}$

Ex $X =$ 

$\tilde{X}$ = universal cover of X = infinite 4-valent tree



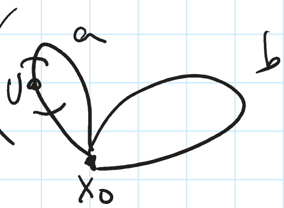
Map:

$p: \tilde{X} \to X$: all vertices $\longrightarrow$ single vertex of X

edges labeled by a $\longrightarrow$ a (with correct orientation)

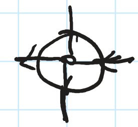
edges labeled by b $\longrightarrow$ b (with correct orientation)

- Clearly continuous (continuous on every edge)
- Covering map:

$p^{-1}\Big($  $\Big)$ = disjoint union of open intervals in $\tilde{X}$

$p^{-1}(U)$ = infinite number of

$p^{-1}(U)$ = infinite number of



$\Rightarrow$ covering map

- $\tilde{X}$ connected and contractible (it's a tree!)

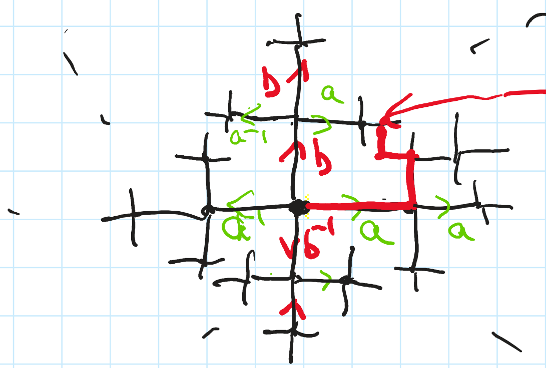
  $\Rightarrow \pi_1(\tilde{X}) = \{e\}$.

<u>Conclusion:</u> bijection $p^{-1}(x_0) \longleftrightarrow \pi_1(X)$

vertices of infinite tree

words in a, b, a^{-1}, b^{-1}

$\Rightarrow$ $\pi_1(X)$ is a free group in a, b.



$aba^{-1}b$