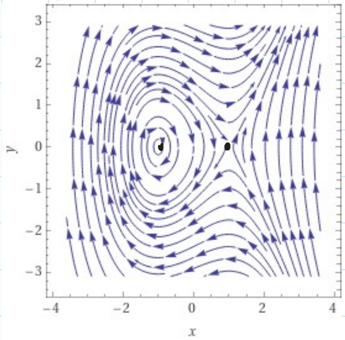
Lecture 18 (2/27)

Monday, February 27, 2023 7:59 AM

Integral curves



From <https://www.wolframalpha.com/input?i=plot+vector+field+%28y%2Cx%5E2-1%29>

$V(x,y) = (y, x^2-1)$

Singular points: $y=0$, $x^2-1=0$
$x = \pm 1$

$(1,0)$ and $(-1,0)$

Linearize near singular points:

$$J = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x & 0 \end{pmatrix}$$

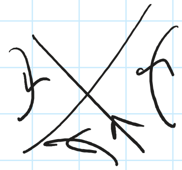
Near $(1,0)$ $\quad x^2-1 \approx \underbrace{2}_{\text{derivative at } x=1}(x-1)$

$v \sim (y, 2(x-1))$

$J = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ eigenvalues $\pm\sqrt{2}$

one positive, one negative eigenvalue

$\Rightarrow$ saddle



Near $(-1,0)$ $\quad V(x,y) \sim (y, -2(x+1))$

$J = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ eigenvalues $= \pm\sqrt{2}\, i$

$\Rightarrow$ center

<u>Note</u> "Standard saddle" $(y,x) = V$ index $= -1$

For <u>any</u> saddle, can locally change coordinates to make it look like this

"Standard center" $(y,-x) = V$ index $= 1$.

Covering Spaces: X − path connected locally path connected

Covers

Recap: X = path connected, locally path connected, semilocally simply connected $\Leftarrow$ connected CW complex

$$\tilde{X} = \{ [\gamma] : \gamma = \text{path in } X \text{ starting at } x_0 \}$$

Last time: defined topology on $\tilde{X}$, and $p : \tilde{X} \to X$

Thm $\tilde{X}$ is a universal cover of X, that is, $\pi_1(\tilde{X}) = \{e\}$, connected.

Thm (1.36) For any subgroup $H \subset \pi_1(X)$ there exists a cover $p_H : X_H \to X$ such that X_H is connected and $\pi_1(X_H) = p_{H*}(\pi_1(X_H)) = H$.

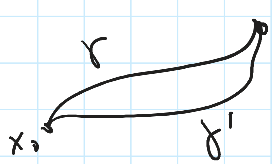
since p_{H*} injective

Proof: $\tilde{X}$ = a universal cover as above

$X_H = \tilde{X}/\sim_H$ where $[\gamma] \sim_H [\gamma']$ if $\gamma(1) = \gamma'(1)$

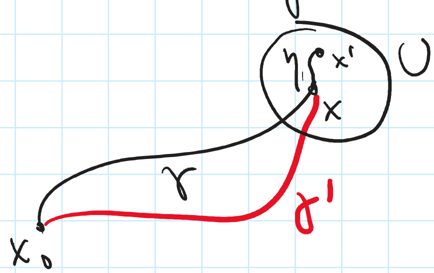
and $[\gamma' \circ \bar{\gamma}] \in H$

loop in X



Exercise H is a subgroup $\Rightarrow$ this is an equivalence relation.

- Need to prove that X_H is a covering of X



U = neighborhood of x

$x' \in U$, can connect it to x by a path η

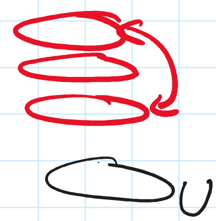
$$U_{[\gamma]} = \{ [\gamma \cdot \eta] : \eta \text{ connects } x \text{ with } x' \in U \}$$

$$U_{[\gamma']} = \{ [\gamma' \circ \eta] : \text{———} / \text{———} \}$$

If $\gamma \underset{H}{\sim} \gamma'$ then $(\gamma' \circ \eta)(\overline{\gamma \cdot \eta}) = \gamma' \eta \bar{\eta} \bar{\gamma} \simeq \gamma' \bar{\gamma} \in H$

so $[\gamma' \eta] \underset{H}{\sim} [\gamma \eta]$ so all elements of $U_{[\gamma]}$ are equivalent to elements of $U_{[\gamma']}$.

So: we identify some neighborhoods in $p^{-1}(U) \subset \tilde{X}$ and still get a covering of X



- Need to check $\pi_1(X_H) = H$

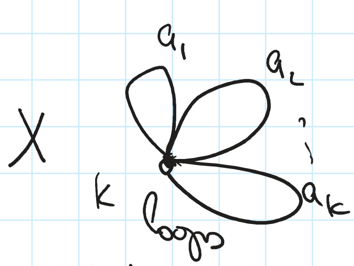
$\pi_1(X_H) = \{$loops in X which lift to loops in $X_H\}$.

γ lifts to a loop in X_H iff $[\gamma] \underset{H}{\sim} [e]$

iff $\gamma \in H$. ∎

Corollary Any subgroup of a free group is free (!)
(finitely generated)

Proof $F = \langle a_1 \ldots a_k \rangle$ free group $\supset H$ subgroup



By thm, there is a covering space $p: X_H \longrightarrow X$ such that $\pi_1(X_H) = H$.

$\pi_1(X) = F$

Any cover of a graph is a graph

(1-dimensional CW complex).

Since X_H is a connected graph, we can contract edges until it has one vertex $\Rightarrow \pi_1(X_H)$ is free. $\boxtimes$

Thm (1.33–1.34) "Lifting property for maps".

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

Some cover $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$

Y path connected, locally path connected

$f : (Y, y_0) \longrightarrow (X, x_0)$ continuous.

We can lift f to a map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $f = p \circ \tilde{f}$ iff

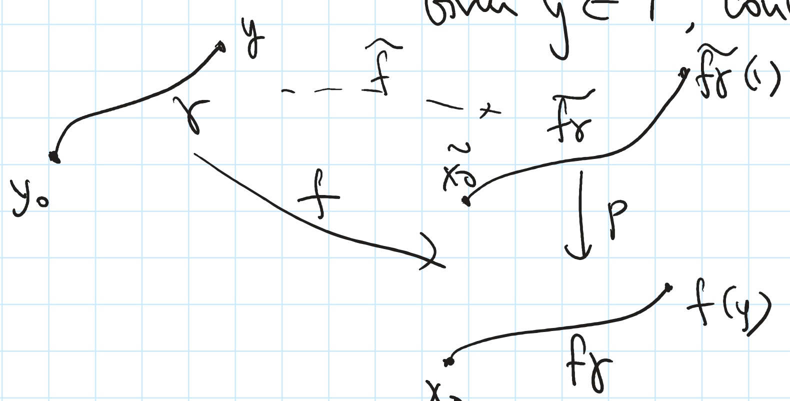
$$f_*(\pi_1(Y, y_0)) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0).$$

Equivalently, $\operatorname{Im} f_* \subset \operatorname{Im} p_*$.

Proof (1) If $\tilde{f}$ exists then $f_* = p_* \tilde{f}_* \Rightarrow \operatorname{Im} f_* \subset \operatorname{Im} p_*$ ✓.

(2) Suppose $\operatorname{Im} f_* \subset \operatorname{Im} p_*$, we need to construct $\tilde{f}$.

Given $y \in Y$, connect y to y_0 by some path γ.



$f\gamma$ is a path in X connecting x_0 to $f(y)$

$\Downarrow$

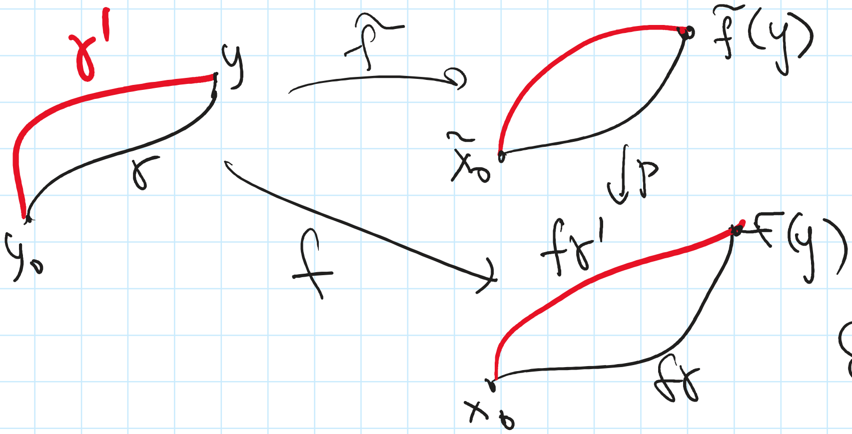
we can lift it to a path in $\tilde{X}$ $\widetilde{f\gamma}$

Define $\boxed{\tilde{f}(y) := \widetilde{f\gamma}(1)}$

Claim: this does not depend on choice of γ

and f is continuous in y.

- Suppose we have two different γ, γ' connecting y_0 and y



<u>Note</u>: $\underbrace{\gamma'\bar{\gamma}}_{h}$ = loop in Y

$(f\gamma')(\overline{f\gamma})$ = loop in X

$= f_* h$

Since $f_* h$ is in the image of p_*, this loop in X lifts to a loop in the cover

$\Rightarrow$ lifts of $f\gamma$ and $'$ meet at same point in $\tilde{X}$.

$\Rightarrow$ $\tilde{f}$ is well defined. $(\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1))$

<u>Exercise/see Hatcher</u>: $\tilde{f}$ is continuous.

∎