Big thms from last 2 lectures

Thm 1 (Existence of covers) X = path conn., locally path conn. semilocally simply conn. ($\Leftarrow$ connected CW complex)

(a) There exists a universal cover $\tilde{X} \longrightarrow X$

$\tilde{X}$ path conn., $\pi_1(\tilde{X}) = \{e\}$.

(b) For any subgroup $H \subset \pi_1(X)$ there exists a cover $p : X_H \longrightarrow X$ such that

$$\pi_1(X_H) \underset{p_* \text{ injective}}{\cong} p_*\pi_1(X_H) = H \subset \pi_1(X)$$

p_* injective.

Thm 2 (Lifting property for maps)

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

$p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ covering

$f : (Y, y_0) \longrightarrow (X, x_0)$ continuous (Y path connected, locally path conn, semilocally simply conn)

The lift $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $f = p \circ \tilde{f}$ exists iff $f_*\pi_1(Y) \subset p_*\pi_1(\tilde{X})$

If $\tilde{f}$ exists, it is unique.

Thm (Classification of covers with basepoints) There is a bijection

$$\left\{ \begin{array}{c} \text{subgroups of} \\ \pi_1(X) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Connected covers } \tilde{X} \to X \\ \text{up to isomorphism} \end{array} \right\}$$

$$\left\{ \quad \pi_1(X) \quad \right\} \longleftrightarrow \left\{ \text{up to isomorphism} \right\}$$

p, p' covers.

Rmk Equivalence relation on covers

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & (\tilde{X}', \tilde{x}_0') \arrow[dl, "p'"] \\
& (X, x_0) &
\end{tikzcd}$$

φ = homeomorphism such that $\varphi(\tilde{x}_0) = \tilde{x}_0'$

and $p' \circ \varphi = p$. $\quad (\Longleftrightarrow p \circ \varphi^{-1} = p')$

Proof Given a cover $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$

get a subgroup $p_* \pi_1(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$

Clear: Isomorphic/equivalent covers $\longrightarrow$ same subgroup

Surjective: Thm 1 (for any subgroup there is a connected cover)

Injective:

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[rr, dashed, bend left, "\varphi"] \arrow[dr, "p"'] & & (\tilde{X}', \tilde{x}_0') \arrow[ll, dashed, "\psi"] \arrow[dl, "p'"] \\
& (X, x_0) &
\end{tikzcd}$$

Assume $p_* \pi_1(\tilde{X}) = p'_* \pi_1(\tilde{X}')$

By Thm 2 we can lift (unique) p to a map $\varphi : (\tilde{X}, \tilde{x}_0) \longrightarrow (\tilde{X}', \tilde{x}_0')$

$p' \circ \varphi = p$

By Thm 2 we can find $\psi : (\tilde{X}', \tilde{x}_0') \longrightarrow (\tilde{X}, \tilde{x}_0)$

such that $p \circ \psi = p'$

$\varphi \circ \psi$ and $\psi \circ \varphi$ are lifts of $\mathrm{Id}_X \Rightarrow \varphi \circ \psi = \mathrm{Id}_{\tilde{X}'}$,
$\psi \circ \varphi = \mathrm{Id}_{\tilde{X}}$.

Rmk For subgroups $G \subset H \subset \pi_1(X, x_0)$

$$\begin{tikzcd}
(X_H, x_0^H) \arrow[dr, "p"'] & & (X_G, x_0^G) \arrow[ll, "\varphi"'] \arrow[dl, "p'"] \\
& (X, x_0) &
\end{tikzcd}$$

$p_* \pi_1(X_H) = H$
$\cup$
$p'_* \pi_1(X_G) = G$

By Thm 2 $\exists \varphi : (X_G, x_0^G) \longrightarrow (X_H, x_0^H)$

$p \circ \varphi = p'$

One can check that φ is a covering map

degree $\varphi = |H/G|$

Corollary $\tilde{X}$ = universal cover $\pi_1(\tilde{X}) = \{e\} \subset H$ for any H

$$\begin{tikzcd}
(X^H, x_0^H) \arrow[dr] & & (\tilde{X}, \tilde{x_0}) \arrow[ll, "\text{covering}"'] \arrow[dl] \\
& (X, x_0) &
\end{tikzcd}$$

$\Rightarrow \tilde{X}$ covers any other cover X^H.

"Universal property of the universal cover".

Ex (prelim fall 22) Are there covering maps

(a) $S^2 \to T^2$

(b) $T^2 \to S^2$

(c) $T^2 \to T^2$

Solution: (b) $\pi_1(T^2) \to \pi_1(S^2)$ injective.

Solution: (b) $\pi_1(T^2) \longrightarrow \pi_1(S^2)$ injective.

$$\underset{\mathbb{Z}^2}{\pi_1(T^2)} \qquad \underset{\{e\}}{\pi_1(S^2)}$$

contradiction.

(a) $\pi_1(S^2) \longrightarrow \pi_1(T^2)$ If cover $p: S^2 \to T^2$

$\pi_1(S^2) = \{e\}$, $\pi_1(T^2) = \mathbb{Z}^2$ exists then S^2 would be a universal cover of T^2.

We have universal cover $\mathbb{R}^2 \longrightarrow T^2$

By today's thm, universal cover is unique up to homeomorphism, $\mathbb{R}^2$ is not homeo to S^2.

(c) $id: T^2 \to T^2$

$$\begin{array}{c} T^2 = S^1 \times S^1 \\ p \big\downarrow (n, 1) \text{ degrees.} \\ S^1 \times S^1 \end{array}$$

$\Rightarrow$ degree n cover of $T^2 \to T^2$.

$$p_* \pi_1(T^2) = p_* \mathbb{Z}^2 = \left\langle \begin{matrix} (n,0) \\ (0,1) \end{matrix} \right\rangle \subset \mathbb{Z}^2 = \pi_1(T^2)$$

$$p_*(1,0) = (n,0) \qquad p_*(0,1) = (0,1)$$

Q: Classify all connected covers of S^2

$\pi_1(S^2) = \{e\} \Rightarrow$ only subgroup $H = \{e\}$

cover $= S^2 \xrightarrow{id} S^2$.

Ex Σ = compact orientable connected surface (2-manifold)

<u>Classification of surfaces</u> $\Rightarrow \Sigma = \Sigma_g$ genus g surface for some g.

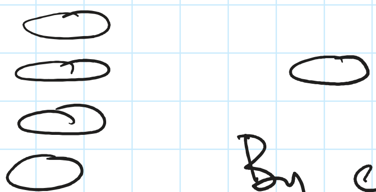
$$\pi_1(\Sigma) = \frac{\langle a_1, b_1, \ldots a_g, b_g \rangle}{\langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = e \rangle}$$

Finite index subgroups of $\pi_1(\Sigma)$

$\updownarrow$

finite connected covers of Σ

$p: M \longrightarrow \Sigma$ finite cover $\Rightarrow M$ compact, connected, oriented surface.



Orientation on $\Sigma \Rightarrow$ orientation on M

By classification, M must be a genus g' surface for some g'

Cor Any finite index subgroup of $\pi_1(\Sigma)$ is isomorphic to $\pi_1(\Sigma_{g'})$ for some g'

Fact (next time) $\Sigma_{g'} \longrightarrow \Sigma_g$ the cover exists iff $g-1$ divides $g'-1$.