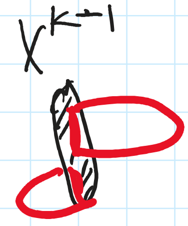
$X$ = cell complex (CW complex)

$X^k$ = $k$-skeleton of $X$ = union of cells of dim $\leq k$




$X^0$ = discrete set of points

$X^k = X^{k-1} \sqcup D^k_\alpha / \sim$

$k$-cells attached along maps

$\phi_\alpha : \partial D^k_\alpha \longrightarrow X^{k-1}$

Most of the time: finitely many cells of each dimension

<u>Warning</u>: if infinitely many cells, topology is a bit subtle, see Hatcher. (Appendix A)

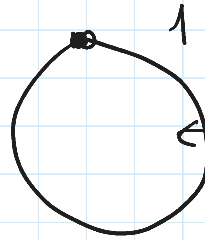
- $Z \subset X$ is closed $\iff$ $Z \cap D^k_\alpha$ is closed
  intersection w. all cells.

- <u>Fact</u>: A cell complex is <u>compact</u> iff it has <u>finitely many cells</u>.

- Given a top. space $X$, we have tons of different cell decompositions, <u>not unique</u>

**Ex 1** $S^n = D^n / \partial D^n$

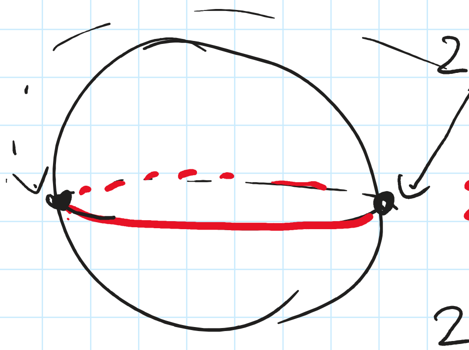
$\phi : \partial D^n \longrightarrow \cdot$ attaching map



1 0-cell = $\{N\}$

1 n-cell = $D^n$

**Ex 1a** $S^2$



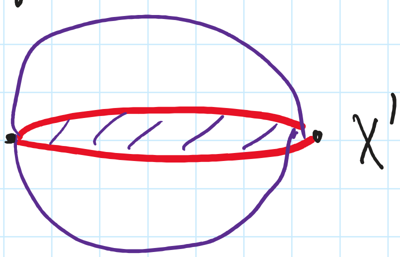
2 0-cells

2 1-cells

2 2-cells : upper/lower hemisphere.

1 - skeleton = equator

**Ex**

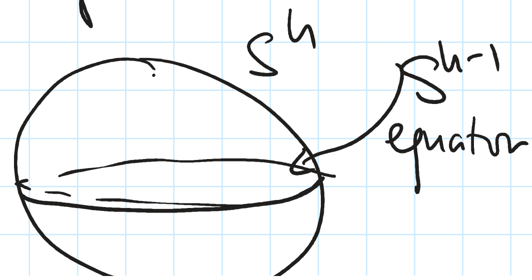



$X^2$ = attach n disks to $X^1$ along boundary.

**Rmk** When we attach k-cells to $X^{k-1}$, they are only allowed to intersect at $X^{k-1}$ but <u>not</u> in the interior.

**Ex 1b** $S^n$ has a cell decomposition with

2 0-cells
2 1-cells
2 2-cells
⋮
2 n-cells



2 - cells

⋮

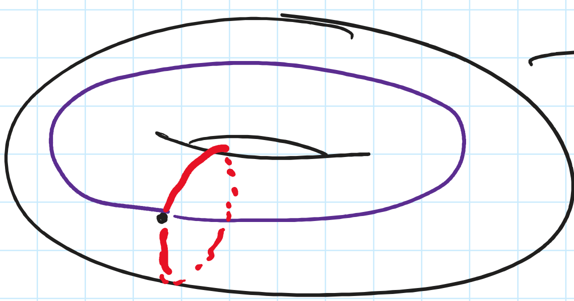
2 n-cells

Given a cell decomp. of $S^{n-1}$, we can attach two $n$-cells on top & bottom

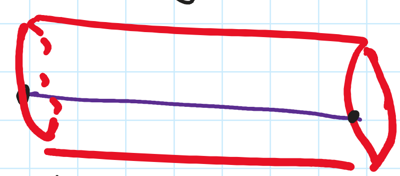
**Def** $S^\infty$ = "infinite-dim. sphere" is cell complex inductively constructed as above, with 2 $k$-cells for *all* $k$.

($k$-skeleton of $S^\infty$) = $S^k$

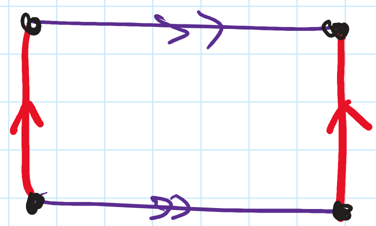
**Ex 2** $T^2$



→ cut along red circle



↙ cut along blue seam



In other words, glue the torus from a square.

• = 0-cell    2 one-cells = red & blue circles

one 2-cell = rectangle $\approx D^2$

Boundary map is complicated!

**Thm** $X, Y$ = cell complexes $\Rightarrow$ $X \times Y$ is a cell complex

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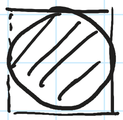
**Proof** $\underset{k\text{-cell in } X}{D^k} \times \underset{\ell\text{-cell in } Y}{D^\ell} \overset{\text{homeo}}{\cong} \underset{\text{cell in } X \times Y}{D^{k+\ell}}$

$D^k \simeq k$-dimensional cube $[-1,1]^k$ (exercise!)



$D^k \times D^\ell \simeq [-1,1]^k \times [-1,1]^\ell = [-1,1]^{k+\ell}$

What happens to the attaching maps?

$$\begin{array}{ccccc} \partial(D^k \times D^\ell) & = & \partial D^k \times D^\ell & \cup & D^k \times \partial D^\ell \\ \downarrow & & \downarrow \phi_X \times Id & & \downarrow Id \times \phi_Y \\ (X \times Y)^{k+\ell-1} & & X^{k-1} \times D^\ell & & D^k \times Y^{\ell-1} \end{array}$$

Exercise: this agrees on the intersection and glues to a continuous map $\partial(D^k \times D^\ell) \to (X \times Y)^{k+\ell-1}$ ∎

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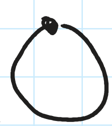
$T^2 = S^1 \times S^1$ $\quad S^1 =$ 

$\Rightarrow$ gives same cell decomposition for $T^2$

Similarly, we construct cell decomposition for $T^n$ = $n$-dim. torus $= S^1 \times S^1 \times \ldots \times S^1$

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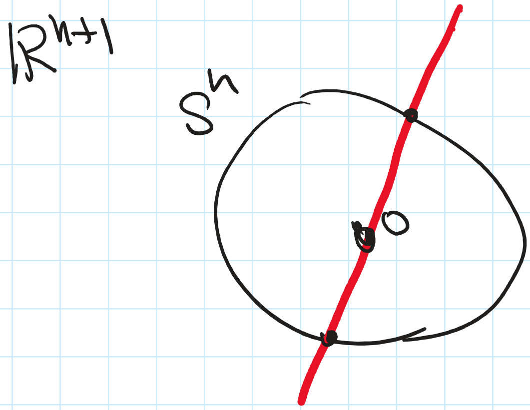
$\mathbb{RP}^n$ = real projective space

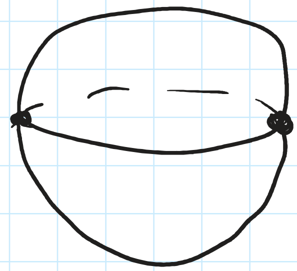
$= \left\{ \begin{array}{c} \text{all lines in } \mathbb{R}^{n+1} \\ \text{through } 0 \end{array} \right\}$

HW

$\simeq S^n/\pm 1 \leftarrow (x_1, \dots, x_n) \sim (-x_1, -x_2, \dots, -x_n)$




**Ex** $\mathbb{R}P^2 = S^2/\pm 1$



← cell decomp of $S^2$ which is invariant under multiplication by $(-1)$.

For $\mathbb{R}P^2$, we choose 1 point in each equivalence class $\Rightarrow$

1 0-cell

1 1-cell

1 2-cell (say, upper hemisphere)

**Q:** How do we attach 2-cell to 1-skeleton?

$\partial(2\text{-cell}) = S^1$ it runs twice along 1-skeleton $=$ circle.

Cannot embed $\mathbb{R}P^2$ in $\mathbb{R}^3$, can embed in $\mathbb{R}^4$.